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PI-S-SYSTEMS

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Abstract. The purpose of this paper is to find some equivalent condition of weakly nonsingular congruence on S-system M_S and this study, we consider p-injective S- system and large subsystem

1. Introduction

Let S be any semigroup. S is called a *semigroup with identity* if S has an element 1 such that $x1 = 1x = x$ for all $x \in S$. An element 0 of S is zero if $x0 = 0x = 0$ for all $x \in S$. If S has no zero element, then it is easy to adjoin an extra element 0 to the set S, and if we define operation on $S \cup \{0\}$ by $0s = s0 = 0$ for all $s \in S \cup \{0\}$ then $S \cup \{0\}$ become a semigroup with zero. We shall consistently use the notation S^0 with the following meaning; $S^0 = S$ if S has a zero element and $S^0 = S \cup \{0\}$ otherwise. Similar to the semigroup S^0, we can define S^1 by $S^1 = S$ if S has an identity element, $S^1 = S \cup \{1\}$ otherwise, then S^1 is a semigroup with identity. Throughout, S will denote a semigroup with or without $1, 0$.

A right S-system M_S over a semigroup S consists of a nonempty set M and a mapping f from $M \times S$ into M, written $f(a, s) = as$ such that for any $a \in M$ and $s, t \in S$ we have $a(st) = (as)t$. A nonempty subset N of M_S is called *S-subsystem* of M_S if and only if it is an S-system with respect to the induced operation. That is $ns \in N$ for all $n \in N, s \in S$. For any nonempty subset B of M_S, $BS = \{bs|b \in B, s \in S\}$ as well as $BS \cup B = BS^1$ are subsets of M_S, and $BS \subset B$ holds if and only if B is S-subsystem of M_S. We omit obvious statements on mS and mS^1 for $m \in M_S$. This concept and some of the following consideration are clear in principle regarding the class of all S-system as a variety of algebras with unary operations, one for each $s \in S$.

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Let M_S and N_S be two S-systems. A mapping $f : M_S \rightarrow N_S$ such that $f(ms) = f(m)s$ for all $m \in M$ and $s \in S$ is called an *S-homomorphism.* The usual definition for monomorphism, epimorphism and isomorphism hold. The set of all S-homomorphism from M_S to N_S is denoted by $Hom_S(M, N)$. If S-system N_S of M_S consist of a single element $\{z\}$, then $zs = z$ for all $s \in S$ and N_S is called *fixed element* of M_S. We denote by $\mathcal{F}(M_S)$ the set of all fixed elements of M_S, which may be empty. By an *centered S-system* M_S over a semigroup S, we mean an S-system containing an unique fixed element. If S has a zero, denoted by 0, then $m0 \in \mathcal{F}(M_S)$ for all $m \in M_S$ and $0 \in S$ and so if M_S is centered S-system, then $\mathcal{F}(M_S) = \{m0\}$ is unique fixed element of M_S and is denoted also 0_M. An S-system M_S is unitary if the semigroup S has an identity 1, and $m1 = m$ for all $m \in M$.

On equationally defined class $\mathcal{C}$, A subalgebra N of M is called *large* in M if and only if for any subalgebra A of $\mathcal{C}$ and any homomorphism f from M to A, with restriction to N is an one to one if f itself is a monomorphism. If N is large in M then we say that M is *essential extension* of N. S-system M_S is called *injective* if and only if for any S-subsystem A_S of S-system B_S and for any S-homomorphism $f : A_S \rightarrow M_S$, there is an S-homomorphism $h : B_S \rightarrow M_S$ such that restriction of h into A_S is f. Indeed, this situation occur and provided on some equational class, such as module over a ring with unit and class of group.

It is well-known that module M_R over a ring R is injective if and only if for any right ideal K of R and for any homomorphism $f : K \rightarrow M$, there exist an element $m \in M$ such that $f(a) = ma$ for all $a \in K$. If we regard ring R as a multiplicative semigroup, then R-module is S-system. But C. V. Hincle, Jr[5] proved that in S-system above result does not hold. So Berthiaume[2] defined weakly injective S-system. An S-system M_S is *weakly injective* if and only if for any right ideal K_S of S_S and for any S-homomorphism $f : K_S \rightarrow M_S$, there exist an element m in M_S such that $f(a) = ma$ for all $a \in K$. For R-module M_R if we think it as R- system, injective is equivalent to weakly injective. But in S-system, injective implies weakly injective, but the converse is not true in general. Also Berthiaume[2] proved that an S-system M_S is injective if and only if it has no proper essential extension.

THEOREM 1.1. *([10], proposition4.4) a) For a semigroup S, each S-system M_S is weakly injective if and only if every right ideal of S has an idempotent generator.*

b) An arbitrary S-system M_S without fixed elements is not injective.

c) If S is a group, each S-system M_S is weakly injective, and M_S is injective if and only if M_S contains a fixed element.

EXAMPLE 1.2. Let G be a group, M_G be any G-system without fixed elements. Then by theorem 1.1 c), M_G is not injective. But M_G is weakly injective. Moreover if we adjoin a fixed elements $\{z\}$ on M_G, then $M \cup \{z\}$ is minimal injective extension of M_G.

Thierrin has called a subsemigroup H of a semigroup S *right reductive* in S if and only if for each $a \neq b$ in S there exists an element $h \in H$ such that $ah \neq bh$. This concept generalized as follows on S-system M_S.

DEFINITION 1.3. Let M_S be an S-system and H a subset of S. Then H is called *reductive* on M_S if and only if for each $a, b \in M, ah = bh$ for all $h \in H$ implies $a = b$.

Generalizing concepts of Feller-Gantos [7] and Hinkle. Jr [5], one defines an singular relation $\psi_S(M)$ on M_S by the set $\{(a, b) \in N \times N | ah = bh$ for all $h \in H$ for some reductive subset H of $S\}$. Clearly we call M_S is non-singular, that is by definition $\psi_S(M) = 1_M$. We call an S-subsystem N_S of M_S is *dense* in M_S if and only if for each $a \neq b, m \in M_S$, there exists an element $s \in S$ such that $as \neq bs$ and $ms \in N_S$.

Regarding right ideal K of a semigroup S as S-subsystem of S_S, we have to speak of a dense right ideal K of S if and only if K_S is dense in S_S. Also Lopez and Ludeman[9] defined for a centered S-system M_S over a semigroup with 0, S-subsystem N_S of S-system M_S is called *meet-large* in M_S if and only if for any nonzero S-subsystem A_S of $M_S, |A \cap N| \neq 0$. It is obvious that subsystem N_S of S-system M_S is meet-large in M_S if and only if $mS^1 \cap N \neq 0$ for any $m \neq 0, m \in M$. For R-module M_R, meet-large submodule and large submodule of M_R are same since there is one to one correspondence between the set of all submodules of M_R and the set of all congruences of M_R.

LEMMA 1.4. *For a centered S-system M_S over a semigroup with 0, if S-subsystem N_S of M_S is large in M_S, then $mS^1 \cap N \neq 0$ for all $m \neq 0 \in M$.*

Proof. Let $\theta = (mS^1 \times mS^1) \cup 1_M$, then θ is not identity congruence since $(m, 0) \in \theta$. Hence there exist an elements $s, t \in M$ such that $s \neq t, (s, t) \in \theta \cap (N \times N)$.

$\theta \cap (N \times N) = [(mS^1 \times mS^1) \cup 1_M] \cap (N \times N) = [(mS^1 \times mS^1) \cap (N \times N)] \cup [(1_M \cap (N \times N)] = [(mS^1 \cap N) \times (mS^1 \cap N)] \cup 1_N$. Thus $|mS^1 \cap N| \geq 2$ and so $mS^1 \cap N \neq 0$ $\square$

From above lemma 1.4, every large subsystem N_S of M_S is meet-large in M_S. But the following example shows that meet-large does not large in general.

EXAMPLE 1.5. Let $S = \{e_0, e_1, e_2, e_3, ...e_n\}, n \geq 3$ be totally ordered set with the order $e_0 < e_1 < e_2 < e_3 < ... <$, then S is finite commutative idempotent semigroup with identity e_n and zero element e_0. We can consider S as S_S-system itself. Let $N_S = \{e_0, e_1, e_2, e_3, ...e_{n-1}\}, f : S \to S$ by $f(e_n) = e_{n-1}, f(e_i) = e_i$ otherwise, then N_S is not large in S_S since $f|_N$ is one to one. For any $e_i \in S, e_iS^1 = \{e_0, e_1, e_2, e_3, ...e_i\}$. Hence if $e_i \neq e_0$, then $\{e_0, e_1\} \subset e_iS^1 \cap N$ contains more than two elements and so N_S is meet-large S-subsystem of S_S.

2. Weakly large S-system

Meet-large system can be defined only on centered S-system, but large and injective S-systems are defined on any S-system. So we defined a weakly large S-system.

DEFINITION 2.1. [7] An S-subsystem N_S of M_S is called *weakly large* if for any non-fixed S-subsystem A_S of $M_S, A_S \cap N_S$ has more than two elements.

For centered S-system M_S, weakly large and meet-large are same. But for non-centered S-system, weakly large and meet-large S-subsystem are different.

LEMMA 2.2. *An S-subsystem N_S of M_S is weakly large in M_S if and only if for any $a \neq b, a, b \in M_S$, there exists $s, t \in S$ such that $as \neq bt$ and $as, bt \in N$.*

Proof. "only if"; The set $aS^1 \cup bS^1$ is S-system of M_S containing more that two elements. So that $|(aS^1 \cup bS^1) \cap N| \geq 2$.

1) If $|aS^1| = 1$ and $|bS^1| = 1$, then a and b are fixed elements of M_S and so $a1 \neq b1, a, b \in N$.

2) $|aS^1| = 1$ and $|bS^1| \geq 2$, then $|aS^1| = 1$ and $|bS^1 \cap N| \geq 2$ and so there exist $bt \in bS^1 \cap N$ such that $a1 \neq bt$ and $a1, bt \in N$.

3) If $|aS^1| \geq 2$, take any element bt in $bS^1 \cap N$, then since $|aS^1| \geq 2$ there exist some $as \in aS^1 \cap N$ such that $as \neq bt$ and $as, bt \in N$.

"if"; Let A be any non fixed S-system of M_S. Since $|A| \geq 2$, take any $a, b \in A, a \neq b$ $as \neq bt, as, bt \in N$ for some $s, t \in S^1$. Hence $|A \cap N| \geq 2$ and so N_S is weakly large in M_S. □

In [7] theorem 2.5, every large S-subsystem of M_S is weakly large in M_S. But if S is an infinite chain with maximal element a, then $N = S - \{a\}$ is weakly large subsystem of S_S which is not large in S_S([7] example 2.8.)

LEMMA 2.3. *If N_S is dense S-subsystems of M_S, then for any $a \neq b, a, b \in M_S$, there exists $s \in S$ such that $as \neq bs$ and $as, bs \in N_S$*

Proof. For $a \neq b, a \in M_S$, there exist $s \in S$ such that $as \neq bs$ and $as \in N$. Again for $as \neq bs, bs \in M$, there exist $t \in S$ such that $a(st) \neq b(st)$ and $b(st) \in N, (as)t \in N$. □

If N_S is weakly large S-subsystem of M_S, for any $m \in M, mS \cap N \neq \emptyset$. But the following example shows the converse is not true.

EXAMPLE 2.4. $S = \{a, b, c, e\}$ with operation a, b are left zero, $ca = cb = cc = a, e$ is identity, then S is semigroup and the sets $\{a, b\}, \{a, c\}$ are S-subsystems of S_S. Since $\{a, b\} \cup \{a, c\} = \{a\}, \{a, b\}$ is not weakly S-systems. But for any $mS, mS \cap \{a, b\} \neq \emptyset$.

DEFINITION 2.5. For S-systems $M_S, H \subset S, K \subset M \times M, T \subset M, J \subset S \times S$.
$\mathcal{L}_M(H) = \{(m, n) \in M \times M | mx = nx$ for all $x \in H\}$.
$\mathcal{R}_S(K) = \{s \in S | as = bs$ for all $(a, b) \in M \times M\}$
$\mathcal{R}_S(T) = \{(a, b) \in S \times S | ta = tb$ for all $t \in T\}$
$\mathcal{L}_M(J) = \{a \in M | am = an$ for all $(m, n) \in J\}$

LEMMA 2.6. *If M_S is S-system, $A \subset T \subset M, I \subset J \subset S \times S$, then*
1) $\mathcal{R}_S(T)$ is right compatible, $\mathcal{R}_S(T) \subset \mathcal{R}_S(A)$ and $T\mathcal{R}_S(T) = 1_M$. If T_S is S system, then is congruence of S.
2) $\mathcal{L}_M(J)$ is S-subsystem of M_S if J is a left compatible. $\mathcal{L}_M(J) \subset \mathcal{L}_M(I)$ and $\mathcal{L}_M(J)J = 1_M$.

Proof. 1) Let $(a, b) \in \mathcal{R}_S(T)$. For any $t \in T$, any $s \in S, t(as) = (ta)s = (tb)s = t(bs)$. Thus $(as, bs) \in \mathcal{R}_S(T)$. Let $(a, b) \in \mathcal{R}_S(T)$, then $ta = tb$ for all $t \in T$. So that $ta = tb$ for all $t \in A$ Thus $(a, b) \in \mathcal{R}_S(A)$. For any $t \in T$, for any $(a, b) \in \mathcal{R}_S(T), ta = tb$. Thus $(ta, tb) = 1_M$. If T_S is S-systems, then $t(sa) = (ts)a = (ts)b = t(sb)$. Thus $(sa, sb) \in \mathcal{R}_S(T)$.
2) Let $a \in \mathcal{L}_M(J), s \in S$, then for any $(m, n) \in J, (sm, sn) \in J$ since J is left compatible. Thus $(as)m = a(sm) = a(sn) = (as)n$ and so $as \in \mathcal{L}_M(J)$. If $a \in \mathcal{L}_M(J)$, then $am = an$ for all $(m, n) \in J$ and so $am = an$ for all $(m, n) \in I$. Thus $a \in \mathcal{L}_M(I)$. For any $a \in \mathcal{L}_M(J)$, for any $(m, n) \in J, am = an$. Hence $\mathcal{L}_M(J)J = 1_M$. □

If we use the notation of above definition for S-system $M_S, H \subset S$ is reductive if and only if $\mathcal{L}_M(H) = 1_M$

Definition 2.7. The relation $\psi_S(M) = \{(a,b) \in M \times M | ax = bx$ for all $x \in H$ for some weakly S-subsystem H_S of $S_S\}$ is called *weakly singular relation* of M_S.

If we denote by $\mathcal{W}_S(M)$ the class of all weakly large of M_S, then we can prove easily that $\mathcal{W}_S(M)$ is full transitive and closed under finite intersection. In fact $\psi_S(M) = \{(a,b) \in M \times M | (a,b) \in \mathcal{L}_M(H), H \in \mathcal{W}_S(S)\} = \cup\{\mathcal{L}_M(H) | H \in \mathcal{W}_S(S)\}$

Theorem 2.8. *For any Ssystems M_S, weakly S-singular relation $\psi_S(M)$ of M_S is the set $Z = \{(m,n) \in M \times M | \mathcal{R}_M(m,n) \in \mathcal{W}_S(M)\ \}$*

Proof. For any $(m,n) \in Z, \mathcal{R}_M(m,n) \in \mathcal{W}_S(M)$. Since $(m,n) \in \mathcal{L}_M(\mathcal{R}_M(m,n))$, we have $(m,n) \in \psi_S(M)$. Conversely if $(m,n) \in \psi_S(M)$, there exist some $H \in \mathcal{W}_S(M)$ such that $mx = nx$ for all $x \in H$. So that $(m,n) \in \mathcal{L}_M(H)$ and by lemma 2.6, $H \subset \mathcal{R}_S(\mathcal{L}_M(H)) \subset \mathcal{R}_S(m,n)$. Since H is weakly large S-subsystem of S_S and $\mathcal{W}_S(M)$ is full transitive, $\mathcal{R}_S(m,n) \in \mathcal{W}_S(S)$. □

It is easily seen from above Theorem 2.8, $\psi_S(M)$ is an S-congruence on M_S which is two-sided congruence if $M_S = S_S$. When $\psi_S(M) = 1_M$, we say that M_S is *weakly S-nonsingular* or weakly S-torsion free. It is easy that M_S is weakly S-nonsingular if and only if all elements of $\mathcal{W}_S(S)$ is deductive.

From lemma 2.2, lemma 2.3, dense S-subsystem of any S-system M_S is weakly large S-subsystem, but if S is a semigroup of example 2.4, $E = \{a,b,c\}$ is weakly large S-subsystem of S_S, but it is not dense .

Theorem 2.9. *If semigroup S is weakly S-nonsingular, then every weakly large S-subsystem N_S of S-system M_S is dense.*

Proof. Let N_S be any weakly large S-subsystem of M_S, $a \neq b, n \in M$. Then by [7] corollary 2.12, the set $a^{-1}N = \{x \in S | ax \in N\}$ is weakly large right ideal of S. So that the set $A = a^{-1}N \cap b^{-1}N \cap n^{-1}N$ is nonempty weakly large S-subsystem of S-system S_S and since $a \neq b$, we have $(a,b) \notin \psi_S(M)$. From $A \in W_S(S), \mathcal{L}_M(A) \subset \psi_S(M), (a,b) \notin \mathcal{L}_M(A)$, there exist an element s in A such that $as \neq bs$. Thus $as, bs, ns \in N$ and so N_S is dense S-subsystem of M_S. □

Lopez and Luedeman defined $\gamma(a) = \{x \in S |$ there exists $b \in aS$ such that $bt = xt$ for all $t \in aS\}$

Lemma 2.10. $\gamma(a) = \{x \in S | (am, x) \in \mathcal{L}_M(aS)$ *for some* $m \in S\}$

Proof. $b \in aS$ such that $bt = xt$ for all $t \in aS$ means by putting $b = am, (am)t = xt$ for all $t \in aS$ and it is equivalent to $(am, x) \in \mathcal{L}_M(aS)$. □

Theorem 2.11. *Let S be a commutative semigroup. If S_S is weakly S-nonsingular, then for all $a \in S, \gamma(a)$ is dense in S_S.*

Proof. Let show $\gamma(a)$ is weakly large S-subsystem of S_S for all $a \in S$. Then $\gamma(a)$ is dense from above theorem 2.9. Let J be ideal of S such that $|J| \geq 2$

1) in case $|Ja| = 1$. Put $Ja = \{b\}$ then, since $b \in J$ we have $b = ba = ab \in aS$. For any $j \in J$, any $ak \in aS, (ba)(ak) = bka \in Ja = \{b\}, j(ak) = jka \in Ja = \{b\}$. So that $(ab, j) \in \mathcal{L}_M(aS)$ and so $J \subset \gamma(a)$.

2) in case $|Ja| \geq 2$. There are $j, i \in J$ such that $ia \neq ja$. Put $ia = x$, then any $ak \in aS, (ai)(ak) = x(ak)$. So $i \in \gamma(a)$. Similarly $j \in \gamma(a)$. Thus $|\gamma(a) \cap J| \geq 2$. □

3. PI-S-systems

Definition 3.1. S-systems M_S is *p-injective* if and only if for any $a \in S$ and any S-homomorphism $f : aS^1 \to M$, there exist an element $m \in M$ such that $f(a) = ma$.

We can see that every injective and weakly-injective S-systems is p-injective. But the converse does not hold in generally.

Lemma 3.2. *M_S is p-injective if and only if $\mathcal{L}_M(\mathcal{R}_S(x)) \subset Mx$ for all $x \in S$.*

Proof. Let $a \in \mathcal{L}_M(\mathcal{R}_S(x)), h : xS^1 \to M, h(xs) = as$ for all $s \in S^1$. If $xs = xt, s, t \in S^1$ then, $(s, t) \in \mathcal{R}_S(x)$ and so $as = at.h((xs)t) = h(x(st)) = a(st) = (as)t = h(xs)t$. So h is well defined S-homomorphism. Since M_S is p-injective, there exist an element $m \in M$ such that $a = h(x1) = mx1 = mx \in Mx$.

Conversely, Let $f : xS^1 \to M$ be any S-homomorphism. If $(s, t) \in \mathcal{R}_S(x)$, then $f(x)s = f(xs) = f(xt) = f(x)t$. So that $f(x) \in \mathcal{L}_M(\mathcal{R}_S(x)) \subset Mx$. Therefore $f(x) = mx$ for some $m \in M$. Thus M_S is p-injective. □

Theorem 3.3. *S is regular semigroup if and only if every S-system M_S over S is p-injective.*

Proof. Let $a \in S$. For identity map $f : aS^1 \to aS^1$, there exist an element $m = ax \in aS^1$ such that $a = ma$. Thus $a = ma = axa$ for some $x \in S$. Conversely, for any S-systems M_S and any principal right ideal aS^1 of S, since S is regular, $aS^1 = eS$ for some idempotent e([3]Lemma 1.13.) For any S-homomorphism $f : eS \to M$, put $f(e) = m$ then $f(es) = f(ees) = mes$ for all $s \in S$. Thus M_S is p-injective. □

It is easy that if e is an idempotent element of S, then $eS = \mathcal{R}_S(T)$ for some $T \subset S \times S$. But for any relation $T \subset S \times S, \mathcal{R}_S(T)$ is not the form $eS, e = e^2$.

THEOREM 3.4. *If S_S is injective and weakly p-nonsingular semigroup then, $\mathcal{R}_S(T)$ is generated by idempotent element of S_S for any relation $T \subset S \times S$.*

Proof. Let $T \subset S \times S, H(\mathcal{R}_S(T))$ be injective hull of $\mathcal{R}_S(T)$, then since S is right self injective, $H(\mathcal{R}_S(T)) \subset S$. For any $n \in H(\mathcal{R}_S(T)), f : S \to H(\mathcal{R}_S(T)), f(x) = nx$ for all $x \in S$, then by [7] theorem 2.11, $f^{-1}(\mathcal{R}_S(T)) \in \mathcal{W}_S(S)$. For any $t \in f^{-1}(\mathcal{R}_S(T)), (a, b) \in T, ant = bnt$. Thus $f^{-1}(\mathcal{R}_S(T)) \subset \mathcal{R}_S(an, bn)$ and since $\mathcal{W}_S(S)$ is full transitive, $\mathcal{R}_S(an, bn) \in \mathcal{W}_S(S)$. Hence $an = bn$ and so $n \in \mathcal{R}_S(T)$. Thus $\mathcal{R}_S(T)$ is injective and by theorem 1.1 b), $\mathcal{R}_S(T)$ has fixed element e. So that $\mathcal{R}_S(T) = eS, e = e^2$. □

LEMMA 3.5. *For any semigroup S, if aS^1 is p-injective, then aS^1 has an idempotent generator.*

Proof. Let take any $b \in aS^1$ such that $bS^1 = aS^1$ and $h : bS^1 \to aS^1, h(bx) = ax$, then there exist $m \in aS^1$ such that $h(b) = mb$. Hence $b = mb$. Since $m \in aS^1, m = at$ for some $t \in S^1$ and so $b = atb$ and since $a = bx$ for some $x \in S^1, b = bxtb$. Put $bxt = e$, then $b = eb$ and e is idempotent. $eS^1 = (bxt)S^1 = atS^1 \subset aS^1 = bS^1 = ebS^1 \subset eS^1$. Thus $bS^1 = eS^1$. □

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