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THE NIELSEN NUMBER ON ASPHERICAL WEDGE

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Abstract. Let X be a finite polyhedron that is of the homotopy type of the wedge of the torus and the surface with boundary. Let $f : X \to X$ be a self-map of X. In this paper, we prove that if the induced endomorphism of $\pi_1(X)$ is K-reduced, then there is an algorithm for computing the Nielsen number $N(f)$.

1. Introduction

Let X be a finite aspherical polyhedron with the homotopy type of the wedge of a torus and a surface with boundary and let f be a self-map of X. The Nielsen number $N(f)$, by its homotopy invariance, provides a lower bound for the minimum number of fixed points over all maps homotopic to f. The Nielsen number is easy to define geometrically, but it is very difficult to compute. See [2] or [5] for the details.

For a given space X, the algebraic properties of fundamental group $\pi_1(X)$ are quite important to compute the Nielsen numbers on it. If $\pi_1(X)$ is a free or a free product group, then it is very difficult to compute the Nielsen number on X. See [8].

In [7], Khamsemanan and Kim estimate the Nielsen number for maps of aspherical spaces including X. The purpose of this paper is to prove that the Nielsen number for many maps of X is explicitly computed which is just estimated in [7]. Our main result is

Theorem 1.1. *If an endomorphism of $\pi_1(X)$ is K-reduced, then there is an algorithm for computing the Nielsen number.*

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2. Preliminaries

For X a wedge of a torus and a surface with boundary, X is homotopy equivalent to a wedge of a torus and a bouquet of k circles and the fundamental group $\pi_1(X)$ is isomorphic to a free product $G * F_k$ where $G = \langle a, b \mid aba^{-1}b^{-1} = 1\rangle$ and $F_k = \langle c_1, c_2, \cdots, c_k\rangle$. Since the Nielsen number is a homotopy type invariant, we assume that $X = T \vee C$ is a wedge of a torus T and a bouquet of k circles C. We denote the intersection by x_0. Let $f : (X, x_0) \to (X, x_0)$ be a self-map of X. To simplify notation, we denote the fundamental group endomorphism induced by a self-map by the same letter as the map. Since X is aspherical, fundamental group information is sufficient to classify self-maps up to homotopy.

THEOREM 2.1 ([7, Theorem 4.1]). *Every endomorphism f of $\pi_1(X)$ satisfies at least one of the following:*

($H1$) *$f(a) = wg_1w^{-1}$ and $f(b) = wg_2w^{-1}$ for some $g_1, g_2 \in G$ and $w \in \pi_1(X)$.*

($H2$) *$f(a) = u^s$ and $f_\pi(b) = u^t$ for some element $u \in \pi_1(X)$ and integers s and t.*

By the commutativity of the Nielsen number, calculating the Nielsen number for a map satisfying ($H2$) is reduced to the calculation of the Nielsen number for the corresponding map of a surface with boundary. See [7] for details. There are some recent works for the calculation of the Nielsen number on surface with boundary (see [4], [6], [9], and [10]).

Now suppose that f is a self-map of X and that the induced endomorphism of f satisfies the condition ($H1$). Then, there is a map $f' : X \to X$ which is (freely) homotopic to f such that $f'(\cdot) = w^{-1}f(\cdot)w$, which satisfies $f'(G) \subseteq G$. For such a map f', since $f'(G) \subseteq G$, we may deform f' by a homotopy so that $f'|_T$, its restriction to T, has a minimal fixed point set and $f'|_{C-\{x_0\}}$, its restriction to $C - \{x_0\}$, has fixed points which correspond to the occurrences of ${c_i}^{\pm 1}$ in $f'(c_i)$, $1 \le i \le k$. The map thus obtained we call the *standard form* of f and denote it also by f. For the rest of the paper, we assume that all maps $f : X \to X$ will be in standard form.

It is well-known that the number of fixed points of $f|_T$ is equal to the Reidemeister number of $f|_T$ that is easily computed from the endomorphism $f|_G$. For each fixed point x_p of f, let α_p be a representative element of the corresponding Reidemeister class of x_p. If x_p is in T, then we may assume that α_p is in G. If x_p is in $C - \{x_0\}$ which corresponds to a word c_i or ${c_i}^{-1}$ in $f(c_i)$, then applying the Wagner's method in [9], we can denote $f(c_i)$ by $\alpha_p c_i \overline{\alpha}_p^{-1}$. Note that we can also calculate α_p using the Fox derivative [3].

Let x_p and x_q be two fixed points of $f : X \to X$. Then x_p and x_q are in the same fixed point class of f if and only if there exists $z \in \pi_1(X) = \pi_1(X, x_0)$ such that

$$z = \alpha_p^{-1} f(z) \alpha_q. \qquad (*)$$

When equation $(*)$ holds, we will say that x_p and x_q are *equivalent*. In order to compute the Nielsen number of f, it is necessary to classify all equivalent fixed point classes. Each class is assigned a non-negative integer as its index and a class is called *essential* if its index is non-zero. The Nielsen number is the number of essential fixed point classes (see [2] or [5] for the details).

For the next section, we introduce some notation for the free product $\pi_1(X) = G * F_k = G * \langle c_1 \rangle * \cdots * \langle c_k \rangle$. The groups G and $\langle c_i \rangle$, $1 \le i \le k$, are called the *free factors* of $\pi_1(X)$. A *reduced sequence* is a sequence of elements $u_1, u_2, \cdots, u_n$ from $\pi_1(X)$ such that $u_i \ne 1$, u_i is in a free factor, and successive u_i, u_{i+1} are not in the same free factor. Each element u of $\pi_1(X)$ can be uniquely expressed as a product $u = u_1 u_2 \cdots u_n$, where $u_1, u_2, \cdots, u_n$ is a reduced sequence, which is called the *reduced form* of u. If $u = u_1 u_2 \cdots u_n$ is reduced, then $u_1, u_2, \cdots, u_n$ are called the *syllables* of u and the *syllable length* $\lambda(u)$ is n. If $u \in \pi_1(X)$ is not reduced , then the reduced form of u is unique. Let $R(u)$ be the reduced form of u. We define the syllable length $\lambda(u)$ of an arbitrary element u in $\pi_1(X)$ by $\lambda(R(u))$.

3. The main theorem

Let $f : X \to X$ be a map in standard form. The induced endomorphism $f : \pi_1(X) \to \pi_1(X)$ satisfies $f(G) \subseteq G$, and thus $\lambda(f(g)) = 1$ for any $g \ne 1$ in G. For each $i = 1, 2, \cdots, k$, let $X_i = f(c_i)$ and $Y_i = X_i^{\pm 1}$. We consider the initial segments of X_i that cancel in the products in the set

$$\{Y_j X_i \mid Y_j \ne X_i^{-1}, 1 \le j \le k\}.$$

Let S_i be the longest such initial segment of X_i. Similarly, let T_i be the longest terminal segment of X_i of those that cancel in the products in the set

$$\{X_i Y_j \mid Y_j \ne X_i^{-1}, 1 \le j \le k\}.$$

If $|X_i| > |S_i| + |T_i|$, where $|w|$ is the word length of a word w, then let $R_i = S_i^{-1} X_i T_i^{-1}$. In this case, we have $X_i = S_i R_i T_i$ as reduced. (S_i

or T_i may be 1.) If $\lambda(R_i) \geq 2$, then we say that X_i has *strong syllable remnant*. This idea is inspired by Wagner's algebraic condition *remnant* for free group endomorphisms [9].

DEFINITION 3.1. Let $f : \pi_1(X) \to \pi_1(X)$ be an endomorphism. We call f *K-reduced* if the following conditions hold:

$(K1)$ for all $i = 1, 2, \cdots, k$ and $g \in G$, $\lambda(gY_i) \geq \lambda(Y_i)$;
$(K2)$ all X_i, $1 \leq i \leq k$, have strong syllable remnant.

The following theorem is the main result in this paper.

THEOREM 3.2. *If an endomorphism $f : \pi_1(X) \to \pi_1(X)$ is K-reduced, then there is an algorithm for computing the Nielsen number $N(f)$ of $f : X \to X$.*

The following technical lemmas will be used in the proof of the main theorem.

LEMMA 3.3. *Suppose that $f : \pi_1(X) \to \pi_1(X)$ is K-reduced. Let x_p and x_q be two fixed points of f in T. If x_p and x_q are in the same fixed point class of f, then they are in the same fixed point class of $f|_T$.*

Proof. Suppose that two fixed points x_p and x_q in T are in the same fixed point class of f. Then, as we mentioned in the previous section, there is a solution $z = z_1 z_2 \cdots z_n$ in $\pi_1(X) = G * F_k$ to the equation

$$z = \alpha_p^{-1} f(z) \alpha_q. \qquad (*)$$

We assume that $z = z_1 z_2 \cdots z_n$ is in reduced form, and thus $\lambda(z) = n$. We will prove that $n = 1$ and $z = z_1$ is in G. Suppose first that z_1 and z_n are not in G. Since f is K-reduced and α_p, α_q are in G, we have $\lambda(\alpha_p^{-1} f(z) \alpha_q) \geq n+2$, and so this is a contradiction. Now suppose that either z_1 or z_n is in G. We assume that $z_1 \in G$ and $z_n \notin G$. The other case is similar. If $f(z_1) \neq \alpha_p$, then since f is K-reduced, $\lambda(\alpha_p^{-1} f(z) \alpha_q) \geq n + 1$, which contradicts to $\lambda(z) = n$. If $f(z_1) = \alpha_p$, then the first letter of $\alpha_p^{-1} f(z) \alpha_q$ is the first letter of $f(z_2)$. Since z_2 is not in G, neither is the first letter of $f(z_2)$ by $(K1)$. But the first letter of z is in G, and thus we have a contradiction. Therefore, z_1 and z_n must be in G. If $n \neq 1$, then z_2 and z_{n-1} are not in G. In this case, we can prove that there is no solution z to the equation $(*)$ using the similar argument as above. Hence, we have $n = 1$ and $z = z_1$ is in G. This implies that z is a solution to the equation

$$z = \alpha_p^{-1} f|_G(z) \alpha_q.$$

This means that x_p and x_q are in the same fixed point class of $f|_T$. □

LEMMA 3.4. *Suppose that $f : \pi_1(X) \to \pi_1(X)$ is K-reduced. Let x_p and x_q be two fixed points of f such that they are not in the fixed point class of x_0. If x_p is in T and x_q is in C, then they are not in the same fixed point class of f.*

Proof. Suppose that two fixed points x_p and x_q are in the same fixed point class of f. Then there is a solution $z = z_1 z_2 \cdots z_n$ to the equation

$$z = \alpha_p^{-1} f(z) \alpha_q \qquad (*)$$

where $z = z_1 z_2 \cdots z_n$ is in reduced form, α_p is in G, and $X_i = \alpha_q c_i \overline{\alpha}_q^{-1}$ for some i. If $z \in G$, then $\alpha_p^{-1} f(z) \in G$. Since $\alpha_q \neq 1$, by $(K1)$, we have $\alpha_p^{-1} f(z) \alpha_q \notin G$. Thus we have $z \notin G$. Then, since f is K-reduced, all R's in $f(z)$ never cancel in the right hand side of $(*)$ except the last one. If $z_n = c_i^{-\ell}$, $\ell \geq 1$, then the last R_i^{-1} in $f(z_n)$ may cancel with α_q in the right hand side of $(*)$.

Case 1 Suppose that all R's in $f(z)$ do not cancel in the right hand side of $(*)$.

If $z_1 \in G$ and $f(z_1) = \alpha_p$, then as in the proof of Lemma 3.3, the first letters of z and $\alpha_p^{-1} f(z) \alpha_q$ are not the same. Otherwise, since $z \notin G$ and no successive z_i, z_{i+1} are in the same free factor, by $(K1)$ and $(K2)$, we have $\lambda(\alpha_p^{-1} f(z) \alpha_q) \geq n + 1$, which is a contradiction. Therefore, we conclude that there is no solution to the equation $(*)$ in this case.

Case 2 Suppose that $z_n = c_i^{-\ell}$, $\ell \geq 1$ and there is a cancellation between the last R_i^{-1} in $f(z_n)$ and α_q in the right hand side of $(*)$.

Let $z' = z c_i$, then $z = z' c_i^{-1}$ and so

$$\begin{aligned} z' c_i^{-1} &= \alpha_p^{-1} f(z' c_i^{-1}) \alpha_q \\ &= \alpha_p^{-1} f(z') X_i^{-1} \alpha_q \\ &= \alpha_p^{-1} f(z') (\overline{\alpha}_q c_i^{-1} \alpha_q^{-1}) \alpha_q \\ &= \alpha_p^{-1} f(z') \overline{\alpha}_q c_i^{-1}. \end{aligned}$$

Therefore, we have $z' = \alpha_p^{-1} f(z') \overline{\alpha}_q$ and since f is K-reduced, all R's in $f(z')$ do not cancel in $\alpha_p^{-1} f(z') \overline{\alpha}_q$. Since x_q and x_0 are not in the same fixed point class, we have $\overline{\alpha}_q \neq 1$ and so we can apply the argument used in Case 1 for the equation $z' = \alpha_p^{-1} f(z') \overline{\alpha}_q$ so that there is no solution z' to the equation $z' = \alpha_p^{-1} f(z') \overline{\alpha}_q$. Consequently, there is no solution to the equation $(*)$. □

Lemma 3.5. *Suppose that $f : \pi_1(X) \to \pi_1(X)$ is K-reduced. Let x_p and x_q be two fixed points of f in C. Then x_p and x_q are in the same fixed point class of f if and only if one of the following holds:*

$$\alpha_p = \alpha_q,\ \alpha_p = \overline{\alpha}_q,\ \overline{\alpha}_p = \alpha_q \text{ or } \overline{\alpha}_p = \overline{\alpha}_q.$$

Proof. Let $X_i = \alpha_p c_i \overline{\alpha}_p^{-1}$ and $X_j = \alpha_q c_j \overline{\alpha}_q^{-1}$. If $\alpha_p = \alpha_q,\ \alpha_p = \overline{\alpha}_q,\ \overline{\alpha}_p = \alpha_q$ or $\overline{\alpha}_p = \overline{\alpha}_q$ hold, then $1, c_j^{-1}, c_i$, or $c_i c_j^{-1}$ are solutions to the equation

$$z = \alpha_p^{-1} f(z) \alpha_q \qquad (*)$$

respectively. Thus x_p and x_q are in the same fixed point class. Conversely, we assume that x_p and x_q are in the same fixed point class, and thus there is a solution $z = z_1 z_2 \cdots z_n$ to the equation

$$z = \alpha_p^{-1} f(z) \alpha_q \qquad (*)$$

where $z = z_1 z_2 \cdots z_n$ is in reduced form. If $z = 1$, then $\alpha_p = \alpha_q$. We now assume that $z \neq 1$.

Case 1 Suppose that neither $z_1 = c_i^{\ell}$, $\ell \geq 1$, and α_p^{-1} cancels with R_i in $\alpha_p^{-1} X_i$ nor $z_n = c_j^{-m}$, $m \geq 1$, and α_q cancels with R_j^{-1} in $X_j^{-1}\alpha_q$.
Without loss of generality, we may assume that $\alpha_q \neq 1$ and thus by $(K1)$, we have $\alpha_q \notin G$. Note that we also have $\alpha_p \notin G$ except the case that $\alpha_p = 1$. Now suppose that $n = 1$. If $z = z_1 \in G$, then $\lambda(z) = 1$ and $f(z) \in G$, but since $\alpha_q \notin G$, by $(K1)$, we have $\lambda(\alpha_p^{-1} f(z) \alpha_q) \geq 2$, which is a contradiction. If $z \in F_k$, then by $(K2)$, we have $\lambda(\alpha_p^{-1} f(z) \alpha_q) \geq 2$, which is contrary to $\lambda(z) = 1$. Thus we conclude that $n \geq 2$. Since no successive z_i, z_{i+1} are in the same free factor, by $(K1)$ and $(K2)$, we have $\lambda(\alpha_p^{-1} f(z) \alpha_q) \geq n+1$, but since $\lambda(z) = n$, we have a contradiction. Consequently, there is no solution to the equation $(*)$ in this case.
Case 2 Suppose that either $z_1 = c_i^{\ell}$, $\ell \geq 1$, and α_p^{-1} cancels with R_i in $\alpha_p^{-1} X_i$ or $z_n = c_j^{-m}$, $m \geq 1$, and α_q cancels with R_j^{-1} in $X_j^{-1}\alpha_q$.
We will show that $\overline{\alpha}_p = \alpha_q$ or $\alpha_p = \overline{\alpha}_q$ in this case. We will only show that $\overline{\alpha}_p = \alpha_q$ under the condition that $z_1 = c_i^{\ell}$, $\ell \geq 1$, and α_p^{-1} cancels with R_i in $\alpha_p^{-1} X_i$. The other case is similar. Suppose that $z_1 = c_i^{\ell}$, $\ell \geq 1$, and α_p^{-1} cancels with R_i in $\alpha_p^{-1} X_i$ and let $z' = c_i^{-1} z$. Then $z = c_i z'$ and

so

$$\begin{aligned} c_i z' &= \alpha_p^{-1} f(c_i z')\alpha_q \\ &= \alpha_p^{-1} X_i f(z')\alpha_q \\ &= \alpha_p^{-1}(\alpha_p c_i \overline{\alpha}_p^{-1}) f(z')\alpha_q \\ &= c_i \overline{\alpha}_p^{-1} f(z')\alpha_q. \end{aligned}$$

Thus we have $z' = \overline{\alpha}_p^{-1} f(z')\alpha_q$. If $z' = 1$, then $\overline{\alpha}_p = \alpha_q$. Otherwise, the argument used in Case 1 does apply for the equation $z' = \overline{\alpha}_p^{-1} f(z')\alpha_q$ and so the equation does not have any solution.
<u>Case 3</u> $z_1 = c_i^{\ell}$ and $z_n = c_i^{-m}$ for some $\ell, m \geq 1$, and α_p^{-1} and α_q cancel with R_i and R_j^{-1} in $\alpha_p^{-1} f(z)\alpha_q$ respectively.
Using the similar argument as in Case 2 in this proof, we can prove that $z = c_i c_j^{-1}$ is the only solution to the equation $(*)$ in which we have $\overline{\alpha}_p = \overline{\alpha}_q$. □

We now present the proof of Theorem 3.2.

Proof. By Lemma 3.4, each fixed point class of f is contained in either $T - \{x_0\}$ or C. By Lemma 3.3, the number of essential fixed point classes that are contained in $T - \{x_0\}$ is equal to $N(f|_T) - 1$ where the Nielsen number $N(f|_T)$ is easily computed from the induced endomorphism $f|_G$ (see [1] or [5]). In order to classify all fixed point classes in C, by Lemma 3.5, we only need to compare all α's and $\overline{\alpha}$'s including $\alpha_0 = \overline{\alpha}_0 = 1$, which correspond to the fixed point x_0. After determining all essential fixed point class in C, we can compute the Nielsen number $N(f)$:

$$N(f) = N(f|_T) - 1 + c$$

where c is the number of essential fixed point classes that are contained in C. □

Next example provides a more precise result concerning Example 4.2 in [7].

Example 3.6. Let $X = T \vee C$ with

$$\pi_1(X, x_0) = \langle a, b, c_1, c_2 \mid aba^{-1}b^{-1} = 1\rangle.$$

The endomorphism induced by f is given by

$$\begin{aligned} f(a) &= c_1^{-1}a^2bc_2a^2b^3c_2^{-1}b^{-1}a^{-2}c_1, \\ f(b) &= c_1^{-1}a^2bc_2a^{-1}bc_2^{-1}b^{-1}a^{-2}c_1, \\ f(c_1) &= ac_1^7a^{-1}c_1b, \\ f(c_2) &= c_1c_2^3. \end{aligned}$$

The standard form f' of f is

$$\begin{aligned} f'(a) &= a^2b^3, \\ f'(b) &= a^{-1}b, \\ f'(c_1) &= c_2^{-1}b^{-1}a^{-2}c_1ac_1^7a^{-1}c_1bc_1^{-1}a^2bc_2, \\ f'(c_2) &= c_2^{-1}b^{-1}a^{-2}c_1^2c_2^3c_1^{-1}a^2bc_2. \end{aligned}$$

Khamsemanan and Kim [7] estimated that $N(f) = N(f') \geq 7$. Let us now compute the Nielsen number $N(f')$ exactly. The Nielsen number $N(f'|_T)$ is

$$N(f'|_T) = |\det(I - F)| = 3$$

where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity matrix and $F = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$ represents the endomorphism of $H_1(T)$ induced by f. Now, we note that the fixed points of f' in C are as follows:

fixed point	α_p	$\overline{\alpha}_p$	index
x_0	1	1	-1
x_1	$c_2^{-1}b^{-1}a^{-2}$	$w^{-1}b^{-1}c_1^{-1}ac_1^{-7}a^{-1}$	1
x_2	$w^{-1}a$	$w^{-1}b^{-1}c_1^{-1}ac_1^{-6}$	1
x_3	$w^{-1}ac_1$	$w^{-1}b^{-1}c_1^{-1}ac_1^{-5}$	1
x_4	$w^{-1}ac_1^2$	$w^{-1}b^{-1}c_1^{-1}ac_1^{-4}$	1
x_5	$w^{-1}ac_1^3$	$w^{-1}b^{-1}c_1^{-1}ac_1^{-3}$	1
x_6	$w^{-1}ac_1^4$	$w^{-1}b^{-1}c_1^{-1}ac_1^{-2}$	1
x_7	$w^{-1}ac_1^5$	$w^{-1}b^{-1}c_1^{-1}ac_1^{-1}$	1
x_8	$w^{-1}ac_1^6$	$w^{-1}b^{-1}c_1^{-1}a$	1
x_9	$w^{-1}ac_1^7a^{-1}$	$w^{-1}b^{-1}$	1
x_{10}	$w^{-1}ac_1^7a^{-1}c_1bc_1^{-1}$	w^{-1}	-1
x_{11}	c_2^{-1}	$w^{-1}c_2^{-3}c_1^{-1}w$	-1
x_{12}	$w^{-1}c_1$	$w^{-1}c_2^{-2}$	1
x_{13}	$w^{-1}c_1c_2$	$w^{-1}c_2^{-1}$	1
x_{14}	$w^{-1}c_1c_2^2$	w^{-1}	1
x_{15}	$w^{-1}c_1c_2^3c_1^{-1}a^2b$	1	1

where $w = c_1^{-1}a^2bc_2$.

Since $\alpha_0 = \overline{\alpha}_{15}$ and $\overline{\alpha}_{10} = \overline{\alpha}_{14}$, this implies that x_0 and x_{15} are in the same fixed point class and so are x_{10} and x_{14}. Since both classes are with index zero, we have $c = 12$. Consequently, we have

$$N(f) = N(f') = N(f'|_T) - 1 + c = 3 - 1 + 12 = 14.$$

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