JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **21**, No. 4, December 2008

COMPACTNESS OF MINIMIZING SEQUENCES FOR THE SOBOLEV TRACE INEQUALITY

Young Ja Park*

Abstract. We construct a minimizing sequence for the Sobolev trace inequality which satisfies compactness in the concentration compactness property.

1. Sobolev trace inequalities and the main theorem

We are concerned with the Sobolev trace inequalities on $\mathbb{R}^{n+1}_+$: for $1 < p < n+1$

$$(1) \qquad \left(\int_{\mathbb{R}^n} |f(x)|^q dx\right)^{p/q} \leq A_{p,q} \left(\int_{\mathbb{R}^{n+1}_+} |\nabla u(x,y)|^p dxdy\right), \frac{1}{q} = \frac{n+1-p}{np},$$

where u is an extension of f to the upper half-space in the trace sense and $A_{p,q}$ is a positive constant independent of the function u. We may replace $W^{1,p}(\mathbb{R}^{n+1}_+)$ by $W^{1,p}(\mathbb{R}^{n+1})$ without loss of generality. It remains *open* to find the *best constant* of (1) in general which is of geometric and analytic importance [2, 3]. The question concerning the best constant is equivalent to the following minimization problem:

$$(2) \qquad \mathbf{I} = \inf \left\{ \mathbf{J}(u) : u \in W^{1,p}(\mathbb{R}^{n+1}), \ \int_{\mathbb{R}^n} |u(x,0)|^q dx = 1 \right\},$$

$$(3) \qquad \mathbf{J}(u) := \int_{\mathbb{R}^{n+1}} |\nabla u(x,y)|^p dxdy.$$

In order to obtain a minimizer for the problem (2), we consider a minimizing sequence (u_k) for (2): that is, a sequence (u_k) satisfying

$$\mathbf{I} = \lim_{k\to\infty} \mathbf{J}(u_k)$$

Received September 09, 2008; Accepted November 20, 2008.
2000 Mathematics Subject Classification: Primary 46E35.
Key words and phrases: Sobolev trace inequality, concentration compactness, minimizing sequence.

with

$$u_k \in W^{1,p}(\mathbb{R}^{n+1}) \text{ and } \int_{\mathbb{R}^n} |u_k(x,0)|^q dx = 1 \text{ for each } k.$$

In this paper, we investigate compactness of a minimizing sequence. In fact, we construct a minimizing sequence having the property which is stated in the following theorem. Hereafter, $B_r(x)$ represents the ball centered at x with radius r in $\mathbb{R}^n$, $\mathbb{R}^{n+1}$, or $\mathbb{R}^N$, which will be clear in the context.

THEOREM 1.1. *There exist a minimizing sequence (u_k) in $W^{1,p}(\mathbb{R}^{n+1})$ and a sequence $(w_k, \tilde{w}_k) \in \mathbb{R}^n \times \mathbb{R}$ satisfying the following compactness property: for any $\varepsilon > 0$, there exists $R > 0$ such that*

$$\int_{[B_R((w_k,\tilde{w}_k))]^C} |\nabla u_k(x,y)|^p dxdy + \int_{[B_R((w_k,\tilde{w}_k))]^C} |u_k(x,y)|^{\frac{(n+1)q}{n}} dxdy + \int_{[B_R((w_k,\tilde{w}_k))]^C \cap \mathbb{R}^n \times \{0\}} |u_k(x,0)|^q dx \;<\; \varepsilon.$$

The main tool for the proof is a concentration compactness lemma, which is stated in the following:

LEMMA 1.2 (Concentration Compactness). *Let (ρ_k) be a sequence of positive functions in $L^1(\mathbb{R}^N)$ satisfying*

$$\int_{\mathbb{R}^N} \rho_k dx \equiv \lambda_k \to \lambda \;\; (\lambda \text{ fixed}).$$

Then there exists a subsequence (ρ_{k_j}) of (ρ_k) satisfying one of the following possibilities:
(i) (**Compactness**) *there exists a sequence (y_j) in $\mathbb{R}^N$ so that for any $\varepsilon > 0$ there exists a radius $R > 0$ such that*

$$\int_{B_R(y_j)} \rho_{k_j}(x)dx \geq \lambda - \varepsilon.$$

(ii) (**Vanishing**) *for any positive real number R,*

$$\lim_{j\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} \rho_{k_j}(x)dx = 0.$$

(iii) (**Dichotomy**) *there exists $\alpha \in (0,\lambda)$ such that for any $\varepsilon > 0$, there exist $j_0 \in \mathbb{N}$ and sequence $(y_j) \in \mathbb{R}^N$ satsfying for $\eta_j := \rho_{k_j}\chi_{B_R(y_j)}$ and $\xi_j := \rho_{k_j}\chi_{[\mathbb{R}^N - B_{R_j}(y_j)]}$,*

$$\|\rho_{k_j} - (\eta_j + \xi_j)\|_{L^1(\mathbb{R}^N)} < \varepsilon,$$

$$\left| \int_{\mathbb{R}^N} \eta_j(x)dx - \alpha \right| \le \varepsilon, \quad \left| \int_{\mathbb{R}^N} \xi_j(x)dx - (\lambda - \alpha) \right| \le \varepsilon,$$

provided that $j \ge j_0$, *and*

$$\mathbf{dist}(\text{supp}\ \eta_j, \text{supp}\ \xi_j) \to \infty \ \text{ as } j \to \infty,$$

where $\mathbf{dist}(A, B) \equiv \inf\{|a - b| : a \in A \text{ and } b \in B\}$.

2. Proof of the main theorem

Consider a minimizing sequence (u_k) of the minimization problem (2). The idea is to show that vanishing and dichotomy occurring for this sequence of functions can be prevented by judicial choice of dilations, so that the concentration compactness lemma will leave us the only option of the compactness. Theorem is proved in two parts. In part I, we manage with appropriate dilations to make a minimizing sequence (u_k) into a new sequence for which the *vanishing*(in Lemma 1.2) is prevented. Then in part II, we prove the dichotomy does not occur on the modified minimizing sequence.

2.1. Part I: The vanishing can be prevented.

We consider a sequence of functions defined by:

$$P_k(x, y) \equiv |\nabla u_k(x, y)|^p + |u_k(x, 0)|^q \otimes \delta_0(y) + |u_k(x, y)|^{\frac{n+1}{n}q},$$

where δ_0 is the Dirac measure at 0. Then we can see that $P_k \ge 0$ and

$$\int_{\mathbb{R}^{n+1}} P_k(x, y)dxdy = L_k \to L \ge \mathbf{I} + 1$$

by the Sobolev embedding theorem. Consider the *concentration function* Q_k *of* P_k defined as

$$Q_k(t) := \sup_{(x,y) \in \mathbb{R}^n \times \mathbb{R}} \int_{B_t((x,y))} P_k(w, s)dwds \quad \text{for } t > 0.$$

Then (Q_k) is a sequence of non-decreasing continuous functions on $\mathbb{R}^+$. For $\sigma > 0$, consider the concentration function Q_k^σ of

$$P_k^\sigma(x, y) \equiv |\nabla u_k^\sigma(x, y)|^p + |u_k^\sigma(x, 0)|^q \otimes \delta_0(y) + |u_k^\sigma(x, y)|^{\frac{n+1}{n}q},$$

where $u_k^\sigma(x, y)$ is defined by

$$u_k^\sigma(x, y) \equiv \sigma^{-\frac{n}{q}} u_k\left(\frac{x}{\sigma}, \frac{y}{\sigma}\right) \text{ for } x \in \mathbb{R}^n \text{ and } y \in \mathbb{R}.$$

Then we can easily observe that $Q_k^\sigma(t) = Q_k(\frac{t}{\sigma})$. So, there is a chance of vanishing occurring. In order to avoid that, we take a sequence (σ_k) of dilations so that $Q_k^{\sigma_k}(1) = \frac{1}{2}$. We can see that

$$\lim_{k\to\infty} \sup_{(x,y)\in\mathbb{R}^n\times\mathbb{R}} \int_{B_R((x,y))} P_k^{\sigma_k}(w,s)dwds \geq \frac{1}{2} \quad \text{for } R \geq 1,$$

since $Q_k^{\sigma_k}(t) \geq \frac{1}{2}$ for $t \geq 1$. The vanishing is prevented by the choice of dilations. We will denote the new minimizing sequence $\left(u_k^{\sigma_k}\right)$ by (u_k).

2.2. Part II: The dichotomy does not occur.

Suppose the dichotomy occurs. Then there exists $\lambda^* \in (0, L) \cup (L, L']$ $(L' = \sup_k\{L_k\})$ such that for any $\varepsilon > 0$ there exist $(w_k, \tilde{w}_k) \in \mathbb{R}^n \times \mathbb{R}$ and R_k, $k = 0, 1, 2, \cdots$ with $R_k > R_0$ (for $k = 1, 2, \cdots$) and $R_k \to \infty$ so that

$$\left| \lambda^* - \int_{B_{R_0}((w_k,\tilde{w}_k))} P_k(x,y)dxdy \right| < \varepsilon,$$

$$\left| (L - \lambda^*) - \int_{[B_{R_k}((w_k,\tilde{w}_k))]^C} P_k(x,y)dxdy \right| < \varepsilon,$$

$$\int_{R_0<|(x,y)-(w_k,\tilde{w}_k)|<R_k} P_k(x,y)dxdy < \varepsilon,$$

$$\begin{aligned}
&\operatorname{supp}\,[P_k\chi_{B_{R_0}((w_k,\tilde{w}_k))}] \subset B_{R_0}((w_k,\tilde{w}_k)),\\
&\operatorname{supp}\,[P_k(1-\chi_{B_{R_k}((w_k,\tilde{w}_k))})] \subset [B_{R_k}((w_k,\tilde{w}_k))]^C,\\
&\textbf{dist}\left(\operatorname{supp}[P_k\chi_{B_{R_0}((w_k,\tilde{w}_k))}],\ \operatorname{supp}[P_k(1-\chi_{B_{R_k}((w_k,\tilde{w}_k))})]\right)\\
&\qquad \geq \textbf{dist}\left(B_{R_0}((w_k,\tilde{w}_k)), [B_{R_k}((w_k,\tilde{w}_k))]^C\right) \to \infty \ \text{ as } \ k \to \infty.
\end{aligned}$$

Consider two functions $\xi, \eta \in C_b^\infty(\mathbb{R}^{n+1})$ satisfying $0 \leq \xi, \eta \leq 1$,

$$\xi(x,y) = \begin{cases} 1 & \text{if } |(x,y)| \leq 1, \\ 0 & \text{if } |(x,y)| \geq 2, \end{cases} \quad \text{and} \quad \eta(x,y) = \begin{cases} 0 & \text{if } |(x,y)| \leq \frac{1}{2}, \\ 1 & \text{if } |(x,y)| \geq 1. \end{cases}$$

We may take R_1 so that $4R_1 \leq R_k$ for $k = 2, 3, \cdots$. Define

$$\xi_k(x,y) \equiv \xi\left(\frac{x-w_k}{R_1}, \frac{y-\tilde{w}_k}{R_1}\right) \text{ and } \eta_k(x,y) \equiv \eta\left(\frac{x-w_k}{R_k}, \frac{y-\tilde{w}_k}{R_k}\right).$$

We look at the following quantity: for k large enough,

$$\begin{aligned}\mathbf{M} &= \int_{\mathbb{R}^{n+1}} |\nabla u_k|^p dxdy - \int_{\mathbb{R}^{n+1}} |\nabla(u_k\xi_k)|^p dxdy - \int_{\mathbb{R}^{n+1}} |\nabla(u_k\eta_k)|^p dxdy \\ &= \int_{B_{R_k}-B_{R_1}} |\nabla u_k|^p dxdy - \int_{B_{2R_1}-B_{R_1}} |\nabla(u_k\xi_k)|^p dxdy \\ &\qquad - \int_{B_{R_k}-B_{\frac{1}{2}R_k}} |\nabla(u_k\eta_k)|^p dxdy \\ &\equiv \mathbf{M}_1 - \mathbf{M}_2 - \mathbf{M}_3.\end{aligned}$$

First, we have

$$\mathbf{M}_1 \leq \int_{R_0<|(x,y)-(w_k,\tilde{w}_k)|<R_k} P_k(x,y)dxdy < \varepsilon.$$

Using Hölder's inequality and the Sobolev embedding theorem together with the assumptions in the beginning of the lemma, we have

$$\begin{aligned}\mathbf{M}_2^{1/p} &\leq \left(\int_{B_{2R_1}-B_{R_1}} |\nabla u_k|^p|\xi_k|^p dxdy\right)^{\frac{1}{p}} + \left(\int_{B_{2R_1}-B_{R_1}} |u_k|^p|\nabla\xi_k|^p dxdy\right)^{\frac{1}{p}} \\ &\leq \left(\int_{B_{R_k}-B_{R_0}} |\nabla u_k|^p dxdy\right)^{\frac{1}{p}} + \left(\int_{B_{2R_1}-B_{R_0}} |\nabla\xi_k|^p|u_k|^p dxdy\right)^{\frac{1}{p}} \\ &< \varepsilon^{1/p} + \left(\int_{\mathbb{R}^{n+1}} |\nabla\xi_k|^{n+1} dxdy\right)^{\frac{1}{n+1}} \left(\int_{B_{R_k}-B_{R_0}} |u_k|^{\frac{n+1}{n}q} dxdy\right)^{\frac{n}{(n+1)q}} \\ &< \varepsilon^{1/p} + C\varepsilon^{\frac{n}{(n+1)q}}.\end{aligned}$$

All the balls in the above inequality are centered at $(w_k, \tilde{w}_k)$. Similarly, we can show that $\mathbf{M}_3^{1/p} < \varepsilon + C\varepsilon^{\frac{n}{(n+1)q}}$. Denote $u_{1k} \equiv u_k\xi_k$, $u_{2k} \equiv u_k\eta_k$. By combining these estimates, we finally have

$$\begin{aligned}|\mathbf{M}| &= \left|\int_{\mathbb{R}^{n+1}} |\nabla u_k|^p dxdy - \int_{\mathbb{R}^{n+1}} |\nabla u_{1k}|^p dxdy - \int_{\mathbb{R}^{n+1}} |\nabla u_{2k}|^p dxdy\right| \\ &< \varepsilon + C\varepsilon^{\frac{np}{(n+1)q}}.\end{aligned}$$

In other words,

$$\begin{aligned}\mathbf{I} &= \lim_{k\to\infty} \int_{\mathbb{R}^{n+1}} |\nabla u_k|^p dxdy \\ &= \lim_{k\to\infty} \int_{\mathbb{R}^{n+1}} |\nabla u_{1k}|^p dxdy + \lim_{k\to\infty} \int_{\mathbb{R}^{n+1}} |\nabla u_{2k}|^p dxdy.\end{aligned}$$

It follows from the assumptions at the beginning that

$$\begin{aligned}\left|\int_{\mathbb{R}^n}|u_{2k}|^q dx-\left(\int_{\mathbb{R}^n}|u_k|^q dx-\int_{\mathbb{R}^n}|u_{1k}|^q dx\right)\right| &\le \int_{B_{R_k((w_k,\tilde{w}_k))}-B_{R_1((w_k,\tilde{w}_k))}}|u_k(x,0)|^q\otimes\delta_0(y)dxdy\\ &\le \int_{R_0\le|(x,y)-(w_k,\tilde{w}_k)|\le R_k}|u_k(x,y)|^q\otimes\delta_0(y)dxdy<\varepsilon.\end{aligned}$$

Let $\alpha_k \equiv \int_{\mathbb{R}^n}|u_{1k}(x,0)|^q dx$, and $\beta_k \equiv \int_{\mathbb{R}^n}|u_{2k}(x,0)|^q dx$. By taking a subsequence, if necessary, we may assume that $\alpha_k\to\alpha$, and $\beta_k\to\beta$. We can see that

$$0\le\alpha,\ \beta\le 1 \text{ and } |\,\beta-(1-\alpha)\,|<\varepsilon.$$

Use the estimates for $\mathbf{M}$ to observe that

$$\left|\int_{\mathbb{R}^{n+1}}|\nabla u_{1k}(x,y)|^p+|u_{1k}(x,y)|^{\frac{(n+1)q}{n}}+|u_{1k}(x,0)|^q\otimes\delta_0(y)dxdy-\lambda^*\right|<\varepsilon,$$

$$\left|\int_{\mathbb{R}^{n+1}}|\nabla u_{2k}(x,y)|^p+|u_{2k}(x,y)|^{\frac{(n+1)q}{n}}+|u_{2k}(x,0)|^q\otimes\delta_0(y)dxdy-(L-\lambda^*)\right|<\varepsilon.$$

We can also see that there is a positive constant γ such that

$$\int_{\mathbb{R}^{n+1}}|\nabla u_{ik}(x,y)|^p dxdy\ge\gamma>0$$

for $i=1,2$ by the Sobolev embedding theorem and the Sobolev trace inequalities together with the estimates above. Now we look at all the possible values for α and β. They are:

$$(a):\alpha_k\to 0\ (\beta_k\to 1),\qquad (b):\alpha\ne 0\ (\beta\ne 1),$$
$$(c):\alpha_k\to 1\ (\beta_k\to 0),\qquad (d):\beta\ne 0\ (\alpha\ne 1).$$

By exchanging the roles of α_k and α with β_k and β, the cases (c) and (d) reduce to the cases (a) and (b). In the case (a), it follows from the estimates for $\mathbf{M}$ that $\mathbf{I}\ge\gamma+\mathbf{I}-\varepsilon$ for all small ε, which leads to a contradiction that $\mathbf{I}\ge\gamma+\mathbf{I}>\mathbf{I}$. For the case (b), we define $\mathbf{I}_\alpha$ as

$$\mathbf{I}_\alpha\equiv\inf\left\{\mathbf{J}(u):\int_{\mathbb{R}^n}|u(x,0)|^q dx=\alpha,\ u\in W^{1,p}(\mathbb{R}^{n+1})\right\},$$

where $\mathbf{J}$ is defined in (3). It easily follows from the definition that $\mathbf{I}=\mathbf{I}_1$ and $\mathbf{I}_\alpha=\alpha^{p/q}\mathbf{I}$. It can also be shown that

$$\mathbf{I}<\mathbf{I}_\alpha+\mathbf{I}_{1-\alpha}\text{ for }0<\alpha<1. \tag{4}$$

Now, in the case (b), we have $\mathbf{I} \geq \mathbf{I}_\alpha + \mathbf{I}_{1-\alpha} - \varepsilon$ for all small $\varepsilon > 0$, which violates (4).

Since we have shown that vanishing and dichotomy can not occur, we complete the proof of the theorem by Lemma 1.2. $\square$

References

[1] W. Beckner, *Inequalities in Fourier analysis*, Ann. Math. **102** (1975), 159-182.

[2] W. Beckner, *Sharp inequalities and geometric manifolds*, J. Fourier Anal. Appl. **3** (1997), 825-836.

[3] *J. F. Escobar, Sharp constant in a Sobolev trace inequality*, Indiana Math. J. **37** (1988), 687-698.

[4] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case, part 1 and 2*, Revista Mathematica Iberoamericana **1** (1985), 145-201, 45-121.

[5] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case, part 1 and 2*, Ann. I. H. P. Anal. Nonlin. **1** (1984), 109-145, 223-283.

*

Department of Mathematics
Hoseo University
Asan Chungnam 336-795, Republic of Korea
E-mail: ypark@hoseo.edu