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ALGEBRAIC STRUCTURES IN A PRINCIPAL FIBRE BUNDLE

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Abstract. Let $P(M, G, \pi) =: P$ be a principal fibre bundle with structure Lie group G over a base manifold M. In this paper we get the following facts:

1. The tangent bundle TG of the structure Lie group G in $P(M, G, \pi) =: P$ is a Lie group.
2. The Lie algebra $\mathfrak{g} = T_eG$ is a normal subgroup of the Lie group TG.
3. $TP(TM, TG, \pi_*) =: TP$ is a principal fibre bundle with structure Lie group TG and projection π_* over base manifold TM, where π_* is the differential map of the projection π of P onto M.
4. for a Lie group H, $TH = H \circ T_eH = T_eH \circ H = TH$ and $H \cap T_eH = \{e\}$, but H is not a normal subgroup of the group TH in general.

1. Introduction

In this note, a general survey on the principal fibre bundle $TP(TM, TG, \pi_*) =: TP$ induced from a principal fibre bundle $P(M, G, \pi) =: P$ which is appeared in [7, p. 55] is explained in details. As by-products, we obtain the following facts:

• the Lie algebra $\mathfrak{g} = T_eG$ is a normal subgroup of the Lie group TG.

• for a Lie group H, $H \circ T_eH = T_eH \circ H = TH$ and $H \cap T_eH = \{e\}$, but the subgroup H of the group TH is not a normal subgroup of TH in general, and so the group TH and the product group $H \times T_eH(= T_eH \times H)$ of H and T_eH are not group-isomorphic in general.

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2. The proof of main results

Let M be a C^∞-manifold and G a Lie group. A principal fibre bundle over M with group G consists of a manifold P and an action of G on P satisfying the following conditions:

(1) G acts freely on P on the right:

$$P \times G \ni (u, a) \mapsto ua = R_a(u) \in P;$$

(2) M is the quotient space of P by the equivalence relation by G, $M = P/G$, and the canonial projection $\pi : P \to M$ is differentiable;

(3) P is locally trivial, that is, every point x of M has a neighbourhood U such that $\pi^{-1}(U)$ is isomorphic with $U \times G$ in the sense that there is a diffeomorphism $\Psi : \pi^{-1}(U) \to U \times G$ such that $\Psi(u) = (\pi(u), \psi(u))$ where ψ is a mapping of $\pi^{-1}(U)$ into G satisfying $\psi(ua) = \psi(u) \cdot a$ for all $u \in \pi^{-1}(U)$ and $a \in G$.

A principal fibre bundle will be denoted by $P(M, G, \pi)$, $P(M, G)$ or simply P. In general, we call P the total space or the bundle space, M the base space, G the structure group and π the projection.

Given a mapping f of a manifold M into another manifold M', the differential at a point $p(\in M)$ of f is the liner mapping f_* of T_pM into $T_{f(p)}M'$ which is defined as follows. For each $X \in T_pM$, choose an integral curve $x(t)$ of the vector X in M such that X is the vector tangent to $x(t)$ at $p = x(t_0)$. Then $f_*(X)$ is the vector tangent to the curve $f(x(t))$ at $f(p) = f(x(t_0))$. It follows immediately that if g is a function differentiable in a neighbourhood of $f(p)$, then $(f_*(X))(g) = X(g \circ f)$. We may also consider f_* as the map of $TM := \bigcup_{p \in M} T_pM$ into $TM' = \bigcup_{q \in M'} T_qM'$.

A binary operation $\circ$ on the tangent bundle TG of a Lie group G can be defined as follows:

For $X \in T_gG$ and $Y \in T_{g'}G$ $(g, g' \in G)$, choose curves $x(t)$ and $y(t)$ in G such that X and Y are the vectors tangent to $x(t)$ and $y(t)$ at $g = x(t_0)$ and $g' = y(t_0)$, respectively. Then $X \circ Y := Xg' + gY := dR_{g'}(X) + dL_g(Y) \in T_{gg'}G \subset TG$ is the vector tangent to the curve $x(t) \cdot y(t)$ at $g \cdot g' = x(t_0) \cdot y(t_0) \in G$.

Then TG is a group with respect to the operation $\circ$ on TG which is just defined. In fact, the zero vector O_e belonging to T_eG is the identity element of TG with respect to the operation $\circ$, where e is the identity element of the Lie group G. For $X \in T_gG, Y \in T_hG$ and $Z \in T_kG$

$(g, h, k \in G)$, $(X \circ Y) \circ Z = X \circ (Y \circ Z)$. Moreover, with respect to the operation $\circ$ on TG, the inverse element of $X (\in T_g G)$ which is tangent to a curve $x(t)$ in G at $g = x(t_0)$ is the vector tangent to the curve $x(t)^{-1}$ at $g^{-1} = x(t_0)^{-1} \in G$.

We may regard the Lie group G and its Lie algerbra $\mathfrak{g}$ also as subgroups $\{O_g | O_g$ is the zero vector in $T_g G, g \in G\}$ and $T_e G$ of the group TG with the operation $\circ$. Moreover, for an arbitrary given $g \in G$ and an arbitrary given X beloing to the space $T_g G$, $X \circ T_e G = T_e G \circ X$. So, $T_e G = \mathfrak{g}$ is a normal subgroup of the Lie group TG.

For $X \in T_g G$, $X \circ G = \{X \circ k = Xk | k \in G\}$ and $G \circ X = \{h \circ X = hX | h \in G\}$. So, in general $X \circ G \neq G \circ X$. Hence, G is not a normal subgroup of the group TG in general. Evidently, $G \cap T_e G = \{e\}$. And, by the definition of the operation $\circ$ on TG, $G \circ T_e G = T_e G \circ G = TG$.

Thus we have

THEOREM 2.1. *Let G be a Lie group. Then,*
(1) *the differential of the group operation of $G \times G$ into G is a group operation on TG.*
(2) *the Lie group G is a subgroup of the Lie group TG.*
(3) *its Lie algebra $\mathfrak{g} = T_e G$ are a normal subgroup of the group TG.*
(4) *$G \circ T_e G = T_e G \circ G = TG$ and $G \cap T_e G = \{e\}$, but in general G is not a normal subgroup of TG.*

Let H and K be two normal subgroups of a group X. Then, X is said to be the *direct product* of the normal subgroups H and K iff $HK (= KH) = X$ and $H \cap K = \{e\}$ ([5, p. 62]).

By virtue of the fact (4) of Theorem 2.1, we obtain

COROLLARY 2.2. *For a Lie group G, $TG = G \circ T_e G = T_e G \circ G = TG$ and $G \cap T_e G = \{e\}$, but the group TG and the product group $G \times T_e G (= T_e G \times G)$ of G and $T_e G$ are not group-isomorphic in general.*

A sequence of group homomorphisms

$$G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \rightarrow \cdots \xrightarrow{f_{n-1}} G_n$$

is said to be *exact* if it is exact at each joint, i.e., if $\mathrm{Im} f_i = \mathrm{Ker} f_{i+1}$ for each $i = 1, 2, \cdots, n-2$. And, we say that an exact sequence

$$G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \rightarrow \cdots \xrightarrow{f_{n-1}} G_n$$

splits at the group $G_i, (i = 2, 3, \cdots, n-1)$, iff the group G_i is the direct product of $\mathrm{Im} f_{i-1} = \mathrm{Ker} f_i$ and another normal subgroup of G_i. Moreover, if an exact sequence splits at each of its non-end groups, we say that it *splits* (or, it is a *split* exact sequence) ([5, p. 71]).

There is the following problem in [7, Problem 4.1 in p. 55]:

Prove the fact that the following exact sequence splits;

$$0 \hookrightarrow \mathfrak{g} = T_eG \hookrightarrow TG \xrightarrow{\pi} G \to 0.$$

Referring to the fact (4) of Theorem 2.1, it may be difficult for us to prove the problem on *split* of the short exact sequence above. Probably, I do not think that the exact sequence above splits in general.

The differential of the action of the group G on $P(M, G, \pi)$ induces the right action of the group TG on TP, (i.e., $TP \times TG \to TP$). Similarly we may regard P as a subset of TP. Moreover, for $A \in \mathfrak{g} \subset TG$ and $u \in P \subset TP, uA(\in TP)$ makes sense. In fact, $uA = A^*_u \in T_uP$, where A^* is the fundamental vector field corresponding to A (cf. [4, 7]). The projection $\pi : P \to M$ and the group operation $u : G \times G \to G$ induce differentials $\pi_* : TP \to TM$ and $\mu_* : TG \times TG = T(G \times G) \to TG$. Here, $\mu_*(= \circ)$ is the group operation on the Lie group TG.

Since P is local trivial, for each point $x \in M$ there exists a proper open neighbourhood U of the point x such that $\pi^{-1}(U)$ is diffeomorphic with $U \times G$. Then there exists a cross section σ_U of U into $\pi^{-1}(U)(\subset P)$. Then $\Psi\rfloor_{\pi^{-1}(U)} : \pi^{-1}(U) \ni u = \sigma_U(\pi(u)) \cdot g_U(\pi(u)) \to (\pi(u), \psi_U(u)) = (\pi(u), g_U(\pi(u)) \in U \times G$ is C^∞-diffeomorphic.

Now, assume that U and V are open neighbourhoods in M with $U \cap V \neq \emptyset$ such that $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are diffeomorphic with $U \times G$, and $V \times G$, respectively. Then $\sigma_V = \sigma_U \varphi_{UV}$ on $U \times V$, where $\varphi_{UV} : U \cap V \longrightarrow G$ is a transition function.

Moreover, $TP \supset \pi_*^{-1}(TU) \supset T_uP \ni B \xrightarrow{\Psi_*} (\pi_*(B), dg_U(c(t))/dt\rfloor_{t=0}) \in TU \times TG$ is C^∞-diffeomorphic, where B is the vector tangent to a curve $\alpha(t)$ at point $u = \alpha(0)$ in P such that $\alpha(t) = \sigma_U(c(t)) \cdot g_U(c(t))$ and $c(t) := \pi(\alpha(t))$. Then, $\sigma_U(c(t))g_U(c(t)) = \sigma_V(c(t))\varphi_{VU}(c(t))g_U(c(t)) = \sigma_V(c(t))g_V(c(t))$ and $dg_V/dt = (d\varphi_{VU}/dt) \cdot g_U + \varphi_{VU} \cdot (dg_U/dt) \equiv (d\varphi_{VU}/dt) \circ (dg_U/dt)$, where the operation $\circ$ is the group operation on TG. Hence the differential $d\varphi_{UV}$ of φ_{UV} is a transition function which is defined on $TU \cap TV$. Thus we obtain

THEOREM 2.3. *Let $P(M, G, \pi)$ be a principal fibre bundle. Then $TP(TM, TG, \pi_*)$ is a principal fibre bundle with group TG over the base manifold TM.*

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