JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 21, No. 2, June 2008

DOUGLAS SPACES OF THE SECOND KIND OF FINSLER SPACE WITH A MATSUMOTO METRIC

Il-Yong Lee*

Abstract. In the present paper, first we define a Douglas space of the second kind of a Finsler space with an (α,β)-metric. Next we find the conditions that the Finsler space with an (α,β)-metric be a Douglas space of the second kind and the Finsler space with a Matsumoto metric be a Douglas space of the second kind.

1. Introduction

The notion of Douglas space was introduced by S. Bácsó and M. Matsumoto [4] as a generalization of Berwald space from viewpoint of geodesic equations. Also, we consider the notion of Landsberg space as a generalization of Berwald space. Recently, the notion of weakly-Berwald space as another generalization of Berwald space was introduced by S. Bácsó and B. Szilágyi [5]. It is remarkable that a Finsler space is a Douglas space if and only if the Douglas tensor ${D_i}^h{}_{jk}$ vanishes identically [6].

The theories of Finsler spaces with an (α,β)-metric have contributed to the development of Finsler geometry [11], and Berwald spaces with an (α,β)-metric have been treated by some authors ([1], [10], [13]).

The purpose of the present paper is to give another different definition of a Douglas space of the Finsler space with an (α,β)-metric, on the basis of the difinition of a Douglas space introduced by M. Matsumoto [12]. Then the Douglas space obtained by a different definition is called a *Douglas space of the second kind*.

This paper is supported by Kyungsung University Research Grant in 2008.
Received March 27, 2008; Accepted May 21, 2008.
2000 *Mathematics Subject Classifications*: Primary 53B40.
Key words and phrases: Berwald space, Douglas space, Dougals space of the second kind, Finsler space, Landsberg space, Matsumoto metric.

Let us define a Douglas space of the second kind. A Finsler space F^n is said to be a Douglas space if $D^{ij} = G^i(x,y)y^j - G^j(x,y)y^i$ are homogeneous polynomials in (y^i) of degree three. Then a Finsler space F^n is said to be a *Douglas space of the second kind* if and only if $D^{im}{}_m = (n+1)G^i - G^m{}_m y^i$ are homogeneous polynomials in (y^i) of degree two. On the other hand, in [12] a Finsler space with an (α,β)-metric is a Douglas space if and only if $B^{ij} = B^i y^j - B^j y^i$ are homegeneous polynomials in (y^i) of degree three. Then a Finsler space of an (α,β)-metric is said to be a *Douglas space of the second kind* if and only if $B^{im}{}_m = (n+1)B^i - B^m{}_m y^i$ are homogeneous polynomials in (y^i) of degree two, where $B^m{}_m$ is given by [8](Theoem 2.1).

The present paper is devoted to defining a Douglas space of the second kind of Finsler space with an (α,β)-metric and studying the condition that a Finsler space of an (α,β)-metric be a Douglas space of the second kind (Theorem 3.1). Next we find the condition that Finsler spaces with a Matsumoto metric $\alpha^2/(\alpha-\beta)$ be a Douglas space of the second kind (Theorem 4.1).

2. Preliminaries

Let $F^n = (M^n, L(\alpha,\beta))$ be said to have an (α,β)-metric, if $L(\alpha,\beta)$ is a postively homogeneous function of (α,β) of degree one, where $\alpha^2 = a_{ij}(x)y^i y^j$ and $\beta = b_i(x)y^i$. The space $R^n = (M^n,\alpha)$ is called the Riemannian space associated with F^n ([2], [11]). In R^n we have the Christoffel symbols $\gamma_j{}^i{}_k(x)$ and the covariant differentiation $(;)$ with respect to $\gamma_j{}^i{}_k$. We shall use the symbols as follows:

$$b^i = a^{ir}b_r, \quad b^2 = a^{rs}b_r b_s,$$

$$2r_{ij} = b_{i;j} + b_{j;i}, \quad 2s_{ij} = b_{i;j} - b_{j;i},$$

$$r^i{}_j = a^{ir}r_{rj}, \quad s^i{}_j = a^{ir}s_{rj}, \quad r_i = b_r r^r{}_i, \quad s_i = b_r s^r{}_i.$$

The Berwald connection $B\Gamma = \{G_j{}^i{}_k, G^i{}_j\}$ of F^n plays one of the leading roles in the present paper. Denote by $B_j{}^i{}_k$ the difference tensor [10] of $G_j{}^i{}_k$ from $\gamma_j{}^i{}_k$:

$$G_j{}^i{}_k(x,y) = \gamma_j{}^i{}_k(x) + B_j{}^i{}_k(x,y).$$

With the subscript 0, transvection by y^i, we have

$$G^i{}_j = \gamma_0{}^i{}_j + B^i{}_j \quad \text{and} \quad 2G^i = \gamma_0{}^i{}_0 + 2B^i,$$

and then $B^i{}_j = \dot{\partial}_j B^i$ and $B_j{}^i{}_k = \dot{\partial}_k B^i{}_j$.

The geodesics of a Finsler space F^n are given by the system of differential equations

$$\ddot{x}^i \dot{x}^j - \ddot{x}^j \dot{x}^i + 2(G^i x^j - G^j x^i) = 0, \quad y^i = \dot{x}^i,$$

in a parameter t. The functions $G^i(x,y)$ are given by

$$2G^i(x,y) = g^{ij}(y^r \dot{\partial}_j \partial_r F - \partial_j F) = \{_j{}^i{}_k\} y^j y^k,$$

where $F = L^2/2$ and $\{_j{}^i{}_k\}$ are Christoffel symbols constructed from $g_{ij}(x,y)$ with respect to x^i.

It is shown [4] that F^n is a Douglas space if and only if the Douglas tensor [6]

$$D_i{}^h{}_{jk} = G_i{}^h{}_{jk} - \frac{1}{n+1}(G_{ijk} y^h + G_{ij}\delta^h_k + G_{jk}\delta^h_i + G_{ki}\delta^h_j)$$

vanishes identically, where $G_i{}^h{}_{jk} = \dot{\partial}_k G_i{}^h{}_j$ is the hv-curvature tensor of the Berwald connection $B\Gamma$ [11].

F^n is said to be a Douglas space [4] if

$$D^{ij} = G^i(x,y)y^j - G^j(x,y)y^i \tag{2.1}$$

are homogeneous polynomials in (y^i) of degree three. Differentiating (2.1) with respect to y^h, y^k, y^p and y^q, we have $D^{ij}_{hkpq} = 0$, which are equivalent of $D^{im}_{hkpm} = (n+1)D_h{}^i{}_{kp} = 0$. Thus if a Finsler space F^n satisfies the condition $D^{ij}_{hkpq} = 0$, which are equivalent to $D^{im}_{hkpm} = (n+1)D_h{}^i{}_{kp} = 0$, we call it a Douglas space. Further differentiating (2.1) by y^m and contacting m and j in the obtained equation, we have $D^{im}{}_m = (n+1)G^i - G^m{}_m y^i$. Thus F^n is said to be a *Douglas space of the second kind* if and only if

$$D^{im}{}_m = (n+1)G^i - G^m{}_m y^i \tag{2.2}$$

are homogeneous polynomials in (y^i) of degree two. Furthermore differentiating (2.2) with respect to y^h, y^j and y^k, we get $D^{im}_{hjkm} = (n+1)D^i_{hjk} = 0$. Therefore we have

Definition 2.1. If a Finsler space F^n satisfies the condition that $D^{im}{}_m = (n+1)G^i - G^m{}_m y^i$ be homogeneous polynomials in (y^i) of degree two, we call it a *Douglas space of the second kind.*

On the other hand, a Finsler space of an (α,β)-metric is said to be a *Douglas space of the second kind* if and only if

$$B^{im}{}_m = (n+1)B^i - B^m{}_m y^i$$

are homogeneous polynomials in (y^i) of degree two, where $B^m{}_m$ is given by [8]. Furthermore differentiating the above with respect to y^h, y^j and y^k. we get $B^{im}_{hjkm} = B^i_{hjk} = 0$. Therefore if a Finsler space F^n with an (α,β)-metric satisfies the condition $B^{im}_{hjkm} = B^i_{hjk} = 0$, we call it a *Douglas space of the second kind.*

Since $L = L(\alpha,\beta)$ is a positively homogeneous function of α and β of degree one, we have

$$\begin{aligned} &L_\alpha\alpha + L_\beta\beta = L, \quad L_{\alpha\alpha}\alpha + L_{\alpha\beta}\beta = 0,\\ &L_{\beta\alpha}\alpha + L_{\beta\beta}\beta = 0, \quad L_{\alpha\alpha\alpha}\alpha + L_{\alpha\alpha\beta}\beta = -L_{\alpha\alpha},\\ &L_\alpha = \partial L/\partial\alpha, \quad L_\beta = \partial L/\partial\beta, \quad L_{\alpha\alpha} = \partial^2 L/\partial\alpha\partial\alpha,\\ &L_{\alpha\beta} = L_{\beta\alpha} = \partial^2 L/\partial\alpha\partial\beta, \quad L_{\alpha\alpha\alpha} = \partial^3 L/\partial\alpha\partial\alpha\partial\alpha. \end{aligned} \tag{2.3}$$

Here we state the following lemma and remark for the later frequent use:

Lemma 2.2 [3]. *If $\alpha^2 \equiv 0 \pmod{\beta}$, that is, $a_{ij}(x)y^iy^j$ contains $b_i(x)y^i$ as a factor, then the dimension is equal to two and b^2 vanishes. In this case we have $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_ib^i = 2$.*

Remark 2.3. Throughout the present paper, we say "homogeneous polynomial(s) in (y^i) of degree r"as $hp(r)$ for brevity. Thus $\gamma_0{}^i{}_0$ is $hp(2)$ and, if the Finsler space with an (α,β)-metric is a Douglas space of the second kind, then $B^{im}{}_m$ is $hp(2)$.

3. Douglas space of the second kind with (α,β)-metric

In the present section, we deal with the condition that a Finsler space with an (α,β)-metric be a Douglas space of the second kind.

Let us consider the function $G^i(x,y)$ of F^n with an (α,β)-metric. According to ([10], [11]), they are written in the form

$$
\begin{aligned}
&2G^i = \gamma_0{}^i{}_0 + 2B^i, \\
&B^i = (E/\alpha)y^i + (\alpha L_\beta/L_\alpha)s^i{}_0 - (\alpha L_{\alpha\alpha}/L_\alpha)C^*\{(y^i/\alpha) - (\alpha/\beta)b^i\},
\end{aligned} \tag{3.1}
$$

where we put

$$
\begin{aligned}
&E = (\beta L_\beta/L)C^*, \\
&C^* = \{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)\}/\{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})\}, \\
&\gamma^2 = b^2\alpha^2 - \beta^2.
\end{aligned}
$$

Since $\gamma_0{}^i{}_0 = \gamma_j{}^i{}_k(x)y^iy^j$ is $hp(2)$, by means of (2.1) and (3.1) we have as follows [12]: A Finsler space F^n with an (α,β)-metric is a Douglas space if and only if $B^{ij} = B^iy^j - B^jy^i$ are $hp(3)$. (2.1) gives

$$
B^{ij} = \frac{\alpha L_\beta}{L_\alpha}(s^i{}_0 y^j - s^j{}_0 y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}C^*(b^iy^j - b^jy^i). \tag{3.2}
$$

Then differentiating (3.2) by y^m and contracting m and j in the obtained equation, we have

$$
\begin{aligned}
&B^{im}{}_m \\
&= \dot\partial_m\left(\frac{\alpha L_\beta}{L_\alpha}\right)(s^i{}_0 y^m - s^m{}_0 y^i) + \frac{\alpha L_\beta}{L_\alpha}\dot\partial_m(s^i{}_0 y^m - s^m{}_0 y^i) \\
&+ \dot\partial_m\left(\frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}\right)C^*(b^iy^m - b^my^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}(\dot\partial_m C^*)(b^iy^m - b^my^i) \\
&+ \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}C^*\dot\partial_m(b^iy^m - b^my^i).
\end{aligned} \tag{3.3}
$$

Making use of (2.2) and the homogeneity of (y^i), we obtain

$$
\dot\partial_m\left(\frac{\alpha L_\beta}{L_\alpha}\right)(s^i{}_0 y^m - s^m{}_0 y^i) = \left(\frac{\alpha L_\beta}{L_\alpha}\right)s^i{}_0 - \frac{\alpha^2 L L_{\alpha\alpha} s_0}{(\beta L_\alpha)^2}y^i, \tag{3.4}
$$

$$
\frac{\alpha L_\beta}{L_\alpha}\dot\partial_m(s^i{}_0 y^m - s^m{}_0 y^i) = \frac{n\alpha L_\beta}{L_\alpha}s^i{}_0, \tag{3.5}
$$

$$\dot{\partial}_m\left(\frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}\right)(b^i y^m - b^m y^i)C^* = \frac{\gamma^2\{\alpha L_\alpha L_{\alpha\alpha\alpha} + (2L_\alpha - \alpha L_{\alpha\alpha})L_{\alpha\alpha}\}C^*}{(\beta L_\alpha)^2} y^i, \tag{3.6}$$

$$(\dot{\partial}_m C^*)y^m = 2C^*, \tag{3.7}$$

$$\begin{aligned}(\dot{\partial}_m C^*)b^m = \frac{1}{2\alpha\beta\Omega^2}\big[&\Omega\{\beta(\gamma^2 + 2\beta^2)M + 2\alpha^2\beta^2 L_\alpha r_0 \\ &- \alpha\beta\gamma^2 L_{\alpha\alpha} r_{00} - 2\alpha(\beta^3 L_\beta + \alpha^2\gamma^2 L_{\alpha\alpha})s_0\} \\ &- \alpha^2\beta M\{2b^2\beta^2 L_\alpha - \gamma^4 L_{\alpha\alpha\alpha} - b^2\alpha\gamma^2 L_{\alpha\alpha}\}\big],\end{aligned} \tag{3.8}$$

$$\frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^* \dot{\partial}_m (b^i y^m - b^m y^i) = \frac{(n-1)\alpha^2 L_{\alpha\alpha} C^*}{\beta L_\alpha} b^i, \tag{3.9}$$

where

$$\begin{aligned}&M = (r_{00} L_\alpha - 2\alpha s_0 L_\beta), \\ &\Omega = (\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}), \quad \text{provided that } \Omega \neq 0, \\ &Y_i = a_{ir} y^r, \quad s_{00} = 0, \quad b^r s_r = 0, \quad a^{ij} s_{ij} = 0.\end{aligned} \tag{3.10}$$

Substituting (3.4), (3.5), (3.6), (3.7), (3.8) and (3.9) into (3.3), we have

$$\begin{aligned}B^{im}{}_m = &\frac{(n+1)\alpha L_\beta}{L_\alpha} s^i{}_0 + \frac{\alpha\{(n+1)\alpha^2 \Omega L_{\alpha\alpha} b^i + \beta\gamma^2 A y^i\}}{2\Omega^2} r_{00} \\ &- \frac{\alpha^2\{(n+1)\alpha^2 \Omega L_\beta L_{\alpha\alpha} b^i + B y^i\}}{L_\alpha \Omega^2} s_0 - \frac{\alpha^3 L_{\alpha\alpha} y^i}{\Omega} r_0,\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}A &= \alpha L_\alpha L_{\alpha\alpha\alpha} + 3L_\alpha L_{\alpha\alpha} - 3\alpha(L_{\alpha\alpha})^2, \\ B &= \alpha\beta\gamma^2 L_\alpha L_\beta L_{\alpha\alpha\alpha} + \beta\{(3\gamma^2 - \beta^2)L_\alpha - 4\alpha\gamma^2 L_{\alpha\alpha}\}L_\beta L_{\alpha\alpha} \\ &\quad + \Omega L L_{\alpha\alpha}.\end{aligned} \tag{3.12}$$

Summarizing up the above, we establish

THEOREM 3.1. *The necessary and sufficient condition for a Finsler space F^n with an (α, β)-metric to be a Douglas space of the second kind is that $B^{im}{}_m$ are homogeneous polynomials in (y^m) of degree two, where $B^{im}{}_m$ is given by (3.11) and (3.12), provided that $\Omega \neq 0$.*

4. Matsumoto space

In the present paper, we consider the condition that Matsumoto space F^n be a Douglas space of the second kind. The notion of this space was originally introduced by M. Matsumoto [9]. The metric of F^n is $L = \alpha^2/(\alpha-\beta)$. Then we get

$$L_\alpha = \alpha(\alpha-2\beta)/(\alpha-\beta)^2, \quad L_\beta = \alpha^2/(\alpha-\beta)^2,$$
$$L_{\alpha\alpha} = 2\beta^2/(\alpha-\beta)^3, \quad L_{\alpha\alpha\alpha} = -6\beta^2/(\alpha-\beta)^4, \tag{4.1}$$
$$\Omega = \alpha\beta^2\{(1+2b^2)\alpha^2 - 3\alpha\beta\}/(\alpha-\beta)^3.$$

Substituting (4.1) into (3.12), we have

$$\begin{aligned} A &= -6\alpha^2\beta^3/(\alpha-\beta)^6, \\ B &= 2\alpha^4\beta^4\{(1-b^2)\alpha^2 - (5+4b^2)\alpha\beta + 9\beta^2\}/(\alpha-\beta)^8. \end{aligned} \tag{4.2}$$

Further substituting (4.1) and (4.2) into (3.11), we get

$$\begin{aligned} &\alpha(\alpha-2\beta)\{(1+2b^2)\alpha-3\beta\}^2 B^{im}{}_m \\ &\quad - (n+1)\alpha^3\{(1+2b^2)\alpha-3\beta\}^2 s^i{}_0 \\ &\quad - (\alpha-2\beta)[(n+1)\alpha^2\{(1+2b^2)\alpha-3\beta\}b^i - 3\gamma^2 y^i]r_{00} \\ &\quad + 2\alpha^2[(n+1)\alpha^2\{(1+2b^2)\alpha-3\beta\}b^i \\ &\quad + \{(1-b^2)\alpha^2 - (5+4b^2)\alpha\beta + 9\beta^2\}y^i]s_0 \\ &\quad + 2\alpha^2(\alpha-2\beta)\{(1+2b^2)\alpha-3\beta\}y^i r_0 = 0. \end{aligned} \tag{4.3}$$

Suppose that F^n be a Douglas space of the second kind, that is, $B^{im}{}_m$ be $hp(2)$. Since α is irrational in (y^i), (4.3) is divided two equations as follows:

$$\begin{aligned} &\alpha^2\{(1+2b^2)^2\alpha^2 + 3(7+8b^2)\beta^2\}B^{im}{}_m + 6(n+1)(1+2b^2)\alpha^4\beta s^i{}_0 \\ &\quad - [(n+1)\alpha^2\{(1+2b^2)\alpha^2 + 6\beta^2\}b^i + 6\beta\gamma^2 y^i]r_{00} \\ &\quad - 2\alpha^2[3(n+1)\alpha^2\beta b^i - \{(1-b^2)\alpha^2 + 9\beta^2\}y^i]s_0 \\ &\quad + 2\alpha^2\{(1+2b^2)\alpha^2 + 6\beta^2\}y^i r_0 = 0, \end{aligned} \tag{4.4}$$

$$\begin{aligned} &\beta\{4(1+2b^2)(2+b^2)\alpha^2 + 18\beta^2\}B^{im}{}_m \\ &\quad + (n+1)\alpha^2\{(1+2b^2)^2\alpha^2 + 9\beta^2\}s^i{}_0 \\ &\quad - \{(n+1)(5+4b^2)\alpha^2\beta b^i + 3\gamma^2 y^i\}r_{00} \\ &\quad - \alpha^2\{2(n+1)(1+2b^2)\alpha^2 b^i - 2(5+4b^2)\beta y^i\}s_0 \\ &\quad + 2(5+4b^2)\alpha^2\beta y^i r_0 = 0. \end{aligned} \tag{4.5}$$

Since only the term $6\beta^3y^ir_{00}$ of (4.4) seemingly does not contain α^2, we must have $hp(4)$ V_4^i such that $\beta^3y^ir_{00} = \alpha^2V_4^i$. First we deal with the general case $\alpha^2 \not\equiv 0 \pmod{\beta}$, that is, $n > 2$. Then there exists a function $f(x)$ such that

$$r_{00} = \alpha^2 f(x); \quad r_{ij} = a_{ij}f(x). \tag{4.6}$$

Transvection by b^iy^j leads to

$$r_0 = \beta f(x); \quad r_j = b_j f(x). \tag{4.7}$$

Since the terms $3\beta^2(6\beta B^{im}{}_m + y^ir_{00})$ of (4.5) seemingly do not contain α^2, there must exist $hp(3)$ $U^i{}_3$ such that

$$3\beta^2(6\beta B^{im}{}_m + y^ir_{00}) = \alpha^2U^i{}_3. \tag{4.8}$$

The above shows there exists $hp(1)$ $U^i = U^i{}_k(x)y^k$ satisfying $U^i{}_3 = \beta^2U^i$, and hence (4.8) is redeced to

$$3(6\beta B^{im}{}_m + y^ir_{00}) = \alpha^2U^i. \tag{4.8$'$}$$

Substituting (4.6) into (4.8′), we have $18\beta B^{im}{}_m = \alpha^2(U^i - 3f(x)y^i)$. Thus from $\alpha^2 \not\equiv 0 \pmod{\beta}$ there exists a function $g^i(x)$ such that $U^i - 3f(x)y^i = 18g^i(x)\beta$, where $g^i = g^i(x)$, which gives

$$B^{im}{}_m = \alpha^2 g^i(x). \tag{4.8$''$}$$

Substituting (4.6) and (4.8″) into (4.4), we have

$$\begin{aligned}\alpha^2\{(1+2b^2)\alpha^2 + 3(7+8b^2)\beta^2\}g^i(x) + 6(n+1)(1+2b^2)\alpha^2\beta s^i{}_0 \\ - f(x)[(n+1)\alpha^2\{(1+2b^2)\alpha^2 + 6\beta^2\}b^i + 6\beta\gamma^2y^i] \\ - 2[3(n+1)\alpha^2\beta b^i - \{(1-b^2)\alpha^2 + 9\beta^2\}y^i]s_0 \\ + 2f(x)\beta\{(1+2b^2)\alpha^2 + 6\beta^2\}y^i = 0.\end{aligned} \tag{4.9}$$

The terms $18\beta^2(f(x)\beta + s_0)y^i$ of (4.9) seemingly do not contain α^2. Thus we can put $18\beta^2(f(x)\beta + s_0)y^i = \alpha^2V^i{}_2$, where $V^i{}_2$ is $hp(2)$. If $V^i{}_2 = h^i(x)\beta^2$, then we have $18(f(x)\beta + s_0)y^i = h^i(x)\alpha^2$. Transvection by b_i

yields $18(f(x)\beta+s_0)=h_b\alpha^2$, where $b_ih^i=h_b$. Thus we obtain $h_b=0$, that is, $f(x)\beta+s_0=0$, which leads to

$$s_0=-f(x)\beta. \tag{4.10}$$

Substituting (4.6), (4.7), (4.8″) and (4.10) into (4.5), we have

$$\begin{aligned}\beta\{4(1+2b^2)(2+b^2)\alpha^2+18\beta^2\}g^i&\\ +(n+1)\{(1+2b^2)^2\alpha^2+9\beta^2\}{s^i}_0&\\ -f(x)\{(n+1)(5+4b^2)\alpha^2\beta b^i+3\gamma^2y^i\}&\\ +f(x)\{2(n+1)(1+2b^2)\alpha^2b^i&\\ -2(5+4b^2)\beta y^i\}\beta+2f(x)(5+4b^2)\beta^2y^i=0.&\end{aligned} \tag{4.11}$$

Only the term $3(6\beta g^i+3(n+1){s^i}_0+f(x)y^i)\beta^2$ of (4.11) seemingly does not contain α^2, and hence we must have $hp(1)$ V^i such that $3(6\beta g^i+3(n+1){s^i}_0+f(x)y^i)\beta^2=\alpha^2V^i$. From $\alpha^2\not\equiv 0 \pmod{\beta}$ it follows that V^i must vanish, and hence

$$3(n+1){s^i}_0=-(6\beta g^i+f(x)y^i). \tag{4.12}$$

Differentiating (4.12) with respect to y^j and transvecting the obtained equation by a_{im}, we have $3(n+1)s_{mj}=-(6g_mb_j+f(x)a_{mj})$, where $a_{im}g^i=g_m$. Hence $3(n+1)(s_{mj}-s_{jm})=-6(g_mb_j-g_jb_m)$, which imply

$$s_{ij}=\frac{1}{n+1}(b_ig_j-b_jg_i). \tag{4.13}$$

Transvection by b^iy^j yields $(n+1)s_0=b^2W-g_b\beta$, where we put $W=g_jy^j$ and $g_b=b^ig_i$. From (4.10) we obtain $b^2W=\{g_b-(n+1)f(x)\}\beta$; $b^2g_j=\{g_b-(n+1)f(x)\}b_j$. Tansvection by b^j leads to $f(x)=0$. Substituting the above into (4.6), we have

$$r_{00}=0;\quad r_{ij}=0. \tag{4.14}$$

Transvecting (4.13) by b^ib^j, we have $(n+1)s_0=b^2W-g_b\beta$. Thus from $s_0=0$, we obtain $b^2W=g_b\beta$.

Conversely substituting $f(x) = 0$, (4.7), (4.10), (4.13) and (4.14) into (4.3), we have $(\alpha - 2\beta)B^{im}{}_m = b^2\alpha^2(b^iW - g^i\beta)$. Transvection by Y_i leads to $B^{0m}{}_m = 0$, that is, $B^{im}{}_m$ is a Douglas space of the second kind.

Next we are concerned with $\alpha^2 \equiv 0 \pmod{\beta}$, that is, Lemma 2.2 shows that $n = 2$, $\alpha^2 = \beta\delta$, $\delta = d_i(x)y^i$, $b^2 = 0$ and $b^id_i = 2$. (4.4) and (4.5) are reduced in the forms respectively

$$\begin{aligned} &\delta(21\beta+\delta)B^{im}{}_m + 18\beta\delta^2 s^i{}_0 - 3\{\delta(6\beta+\delta)b^i - 2\beta y^i\}r_{00} \\ &- 2\delta\{9\beta\delta b^i - (9\beta+\delta)y^i\}s_0 + 2\delta(6\beta+\delta)y^i r_0 = 0, \end{aligned} \tag{4.15}$$

$$\begin{aligned} &2(9\beta+4\delta)B^{im}{}_m + 3\delta(9\beta+\delta)s^i{}_0 - 3(5\delta b^i - y^i)r_{00} \\ &- 2\delta(3\delta b^i - 5y^i)s_0 + 10\delta y^i r_0 = 0. \end{aligned} \tag{4.16}$$

Since only the term $6\beta y^i r_{00}$ of (4.15) seemingly does not contain δ, there must exist $hp(1)$ $V = V_i(x)y^i$ such that

$$r_{00} = \delta V; \quad 2r_{ij} = d_iV_j + d_jV_i. \tag{4.17}$$

Transvection by b^iy^j gives

$$2r_0 = 2V + V_b\delta, \quad V_b = b^iV_i. \tag{4.18}$$

Paying attention to the terms of (4.16) which seeming do not contain δ, we can put

$$18\beta B^{im}{}_m + 3y^ir_{00} = \delta V^i{}_2,$$

where $V^i{}_2$ is $hp(2)$. Substitution by (4.17) leads to

$$B^{im}{}_m = \delta U^i, \tag{4.19}$$

where U^i is $hp(1)$ and $V^i{}_2 - 3y^iV = 18\beta U^i$. Substituting (4.17), (4.18) and (4.19) into (4.15), we obtain

$$\begin{aligned} &\delta(21\beta+\delta)U^i + 18\beta\delta s^i{}_0 - 3\{\delta(6\beta+\delta)b^i - 2\beta y^i\}V \\ &- 2\{9\beta\delta b^i - (9\beta+\delta)y^i\}s_0 + (6\beta+\delta)\,(2V + V_b\delta)\,y^i = 0. \end{aligned} \tag{4.20}$$

Since the terms $18\beta y^i(V+s_0)$ of (4.20) seemingly do not contain δ, there must exist a function $h(x)$ such that

$$s_0 = h(x)\delta - V. \tag{4.21}$$

Further substituting (4.17), (4.18), (4.19) and (4.21) into (4.16), we have

$$\begin{aligned} &2(9\beta+4\delta)U^i + 3(9\beta+\delta)s^i{}_0 - 3\{5\delta b^i - y^i\}V \\ &- 2(3\delta b^i - 5y^i)(h(x)\delta - V) + 5(2V + V_b\delta)y^i = 0. \end{aligned} \tag{4.22}$$

Since the dimension is equal to two and (β,δ) are independant pairs, we can put $V = p(x)\beta + q(x)\delta$ and $U^i = h^i(x)\beta + k^i(x)\delta$ where $p(x)$, $q(x)$, $h^i(x)$ and $k^i(x)$ are scalar functions. Substitution by $V = p(x)\beta + q(x)\delta$ and the terms $3\beta\{6U^i + 9s^i{}_0 + p(x)y^i\}$ of the obtained equation seemingly do not contain δ. Thus there exists function $g^i(x)$ satisfying $9s^i{}_0 + 6U^i + p(x)y^i = \delta g^i(x)$. Paying attention to $U^i = h^i\beta + k^i\delta$, we obtain

$$\begin{aligned} 9s^i{}_0 &= g^i(x)\delta - p(x)y^i - 6h^i(x)\beta; \\ 9s^i{}_j &= g^i(x)d_j - p(x)\delta^i_j - 6h^i(x)b_j. \end{aligned} \tag{4.23}$$

Paying attention to $2a_{ij} = b_i d_j + b_j d_i$ and transvecting (4.23) by a_{im}, we have

$$9s_{ij} = \left\{g_i - \frac{1}{2}p(x)b_i\right\} d_j - \left\{\frac{1}{2}p(x)d_i + 6h_i\right\} b_j, \tag{4.24}$$

where $a_{im}g^i = g_m$ and $a_{im}h^i = h_m$.
Transvecting (4.24) by $b^i y^j$, we have

$$9s_0 = g_b\delta - (p(x) + 6h_b)\beta,$$

where we put $b^i g_i = g_b$ and $b^i h_i = h_b$. Substituting (4.21) into the above, we obtain

$$9V = (9h(x) - g_b)\delta + (p(x) + 6h_b)\beta. \tag{4.25}$$

Summarizing up the above, we obtain

Theorem 4.1. *A Matsumoto space with the metric $L = \alpha^2/(\alpha - \beta)$ is a Douglas space of the second kind, if and only if*

(1) $\alpha^2 \not\equiv 0 \pmod{\beta}$: (4.13) and (4.14) are satisfied, where $b^2W = g_b\beta$,

(2) $\alpha^2 \equiv 0 \pmod{\beta}$: $n = 2$ and (4.17) and (4.24) are satisfied, where $\alpha^2 = \beta\delta$, $\delta = d_i(x)y^i$, and $p(x)$, $h^i(x)$, $g^i(x)$ are scalar functions and V is given by (4.25).

References

[1] T. Aikou, M. Hashiguchi and K. Yamauchi, *On Matsumoto's Finsler space with time measure*, Rep. Fac. Sci. Kagoshima Univ., (Math., Phys. & Chem.), **23** (1990), 1–12.

[2] P. L. Antonelli, R. S. Ingraden and M. Matsumoto, *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology*, Kluwer Acad. Publ., Dordrecht, 1993.

[3] S. Bácsó and M. Matsumoto, *Projective changes between Finsler spaces with (α, β)-metric*, Tensor, N. S. **55** (1994), 252–257.

[4] S. Bácsó and M. Matsumoto, *On Finsler spaces of Douglas type. A generalization of the notion of Berwald space*, Publ. Math. Debrecen, **51** (1997), 385–406.

[5] S. Bácsó and B. Szilágyi, *On a weakly-Berwald space of Kropina type*, Mathematica Pannonica, **13** (2002), 91–95.

[6] L. Berwald, *On Cartan and Finsler geometries*, III, *Two-dimensional Finsler spaces with rectilinear extremal*, Ann. of Math. **42** (1941), 84–112.

[7] M. Hashiguchi, S. Hōjō and M. Matsumoto, *Landsberg spaces of dimension two with (α, β)-metric*, Tensor, N. S. **57** (1996), 145–153.

[8] I. Y. Lee, *On weakly-Berwald spaces of (α, β)-metric*, to appear in J. Korean Math. Soc.

[9] M. Matsumoto, *A slope of a mountain is a Finsler surface with respect to a time measure*, J. Math. Kyoto Univ. **29** (1989), 17–25.

[10] M. Matsumoto, *The Berwald connection of a Finsler space with an (α, β)-metric*, Tensor, N. S. **50** (1991), 18–21.

[11] M. Matsumoto, *Theory of Finsler spaces with (α, β)-metric*, Rep. On Math. Phys. **31** (1992), 43–83.

[12] M. Matsumoto, *Finsler spaces with (α, β)-metric of Douglas type*, Tensor, N. S. **60** (1998), 123–134.

[13] H. S. Park, I. Y. Lee and C. K. Park, *Finsler space with the general approximate Matsumoto metric*, Indian J. pure appl. Math. **34** (2003), no. 1, 59–77.

*
Department of Mathematics
Kyungsung University
Busan 608-736, Republic of Korea
E-mail: iylee@ks.ac.kr