JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **21**, No. 2, June 2008

# PROXIMAL POINT ALGORITHMS BASED ON THE $(A, \eta)$-MONOTONE MAPPINGS

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Abstract. In this paper, we consider proximal point algorithms based on $(A, \eta)$-monotone mappings in the framework of Hilbert spaces. Since $(A, \eta)$-monotone mappings generalize $A$-monotone mappings, $H$-monotone mappings and many other mappings, our results improve and extend the recent ones announced by [R.U. Verma, Rockafellars celebrated theorem based on $A$-maximal monotonicity design, Appl. Math. Lett. 21 (2008), 355-360] and [ R.T. Rockafellar, Monotone operators and the proximal point algorithm, SIAM J. Control Optim. 14 (1976) 877-898] and some others.

## 1. Introduction

Variational inclusions problems are among the most interesting and intensively studied classes of mathematical problems and have wide applications in the fields of optimization and control, economics and transportation equilibrium and engineering sciences. Variational inclusions problems have been generalized and extended in different directions using the novel and innovative techniques. Various kinds of iterative algorithms to solve the variational inequalities and variational inclusions have been developed by many authors, see [1-16].

Let $H$ be a real Hilbert space with the norm $\|\cdot\|$ and the inner product $\langle \cdot, \cdot \rangle$. We consider the classical nonlinear variational inclusion problem: find a solution to

$$0 \in M(x), \tag{1.1}$$

where $M : H \to 2^H$ is a set-valued mapping on $H$. In 1976, Rockafellar [10] investigated the general convergence and rate of convergence based on proximal point algorithms in the context of solving (1.1) by showing,

Received November 06, 2007; Accepted May 20, 2008.
2000 Mathematics Subject Classification: Primary 47H04, 47H05, 49J40.
Key words and phrases: $(A, \eta)$-monotone mapping, $A$-monotone mapping, variational inclusion.

when $M$ is maximal monotone, that the sequence $\{x_n\}$ generated for an initial point $x_0$ by

$$x_{n+1} \approx P_n(x_n) \tag{1.2}$$

converges weakly to a solution (1.1), provided the approximation is made sufficiently accruate as the iteration proceeds, where $P_n = (I + c_n M)^{-1}$ for a sequence $\{c_n\}$ of positive real numbers that is bounded away from zero. It follows from (1.2) that $x_{n+1}$ is an approximate solution to the inclusion problem

$$0 \in M(x) + c_n^{-1}(x - x_n). \tag{1.3}$$

As a matter of fact, a general class of problems of variational character, including minimization or maximization of functions, variational inequality problems, and minimax problems, can be unified into the form (1.1). General maximal monotonicity has been a powerful framework to studying convex programming and variational inequalities. It turned out that one of the fundamental algorithms applied for solving these problems was in fact proximal point algorithm. Furthermore, Rockafellar [11] applied the proximal point algorithm in convex programming. Verma [15] improved the results of Rockafellar [10] based on $A$-maximal monotonicity design. Recently Verma [14] generalized the recently introduced and studied notion of $A$-monotonicity to the case of $(A, \eta)$-monotonicity, while examining the sensitivity analysis for a class of nonlinear variational inclusion problems based on the generalized resolvent operator technique.

Inspired and motivated by the recent research going on in this area, in this paper, we explore the approximation solvability of a generalized nonlinear variational inclusion problem (1.1) based on $(A, \eta)$-monotone mappings in the framework Hilbert spaces. Our results mainly improve and extend the recent ones announced by Rockafellar [10] and Verma [15].

## 2. Preliminaries

In this section we explore some basic properties derived from the notion of $(A, \eta)$-monotonicity. Let $\eta : H \times H :\rightarrow H$ be a single-valued mapping. The map $\eta$ is called $\tau$-Lipschitz continuous if there is a constant $\tau > 0$ such that

$$\|\eta(u, v)\| \leq \tau \|y - v\|, \quad \forall u, v \in H.$$

Let $M : H \to 2^H$ be a multivalued mapping from a Hilbert space $H$ to $2^H$, the power set of $H$. We recall following:

(i) The set $D(M)$ defined by

$$D(M) = \{u \in H : M(u) \neq \emptyset\},$$

is called the effective domain of $M$.

(ii) The set $R(M)$ defined by

$$R(M) = \bigcup_{u \in H} M(u),$$

is called the range of $M$.

(iii) The set $G(M)$ defined by

$$G(M) = \{(u, v) \in H \times H : u \in D(M), v \in M(u)\},$$

is the graph of $M$.

Definition 2.1. Let $\eta : H \times H \to H$ be a single-valued mapping and let $M : H \to 2^H$ be a multivalued mapping on $H$.

(i) The map $M$ is said to be monotone if

$$\langle u^* - v^*, u - v \rangle \geq 0, \quad \forall (u, u^*), (v, v^*) \in G(M).$$

(ii) $\eta$-monotone if

$$\langle u^* - v^*, \eta(u, v) \rangle \geq 0, \quad \forall (u, u^*), (v, v^*) \in G(M).$$

(iii) $r$-strongly monotone if

$$\langle u^* - v^*, u - v \rangle \geq r\|u - v\|, \quad \forall (u, u^*), (v, v^*) \in G(M).$$

(iv) $(r, \eta)$-strongly monotone if

$$\langle u^* - v^*, \eta(u, v) \rangle \geq r\|u - v\|, \quad \forall (u, u^*), (v, v^*) \in G(M).$$

(v) $\eta$-pseudomonotone if $\langle v^*, \eta(u, v) \rangle \geq 0$ implies

$$\langle u^*, \eta(u, v) \rangle \geq 0, \quad \forall (u, u^*), (v, v^*) \in G(M).$$

(vi) $(m, \eta)$-relaxed monotone if there exists a positive constant $m$ such that

$$\langle u^* - v^*, \eta(u, v) \rangle \geq -m\|u - v\|^2, \quad \forall (u, u^*), (v, v^*) \in G(M).$$

Definition 2.2 ([12]). Let $A : H \to H$ be a nonlinear mapping on a Hilbert space $H$ and let $M : H \to 2^H$ be a multivalued mapping on $H$. The map $M$ is said to be $A$-monotone if
(i) $M$ is $m$-relaxed monotone.
(ii) $A + \rho M$ is maximal monotone for $\rho > 0$.

Remark 2.3. $A$-monotonicity generalizes the notion of $H$-monotonicity introduced in [3].

Definition 2.4 ([13],[14]). A mapping $M : H \to 2^H$ is said to be maximal $(m, \eta)$-relaxed monotone if
(i) $M$ is $(m, \eta)$-relaxed monotone,
(ii) for $(u, u^*) \in H \times H$ and

$$\langle u^* - v^*, \eta(u, v)\rangle \geq -m\|u - v\|^2, \quad (v, v^*) \in G(M),$$

we have $u^* \in M(u)$.

Definition 2.5 ([13],[14]). Let $A : H \to H$ and $\eta : H \times H \to H$ be two single-valued mappings. The map $M : H \to 2^H$ is said to be $(A, \eta)$-monotone if
(i) $M$ is $(m, \eta)$-relaxed monotone,
(ii) $R(A + \rho M) = H$ for $\rho > 0$.
Note that alternatively, the map $M : H \to 2^H$ is said to be $(A, \eta)$-monotone if
(i) $M$ is $(m, \eta)$-relaxed monotone,
(ii) $A + \rho M$ is $\eta$-pseudomonotone for $\rho > 0$.

Remark 2.6. $(A, \eta)$-monotonicity generalizes the notion of $A$ - monotonicity introduced by Verma [12] and $(H, \eta)$-monotonicity introduced by Fang et al. [4].

Definition 2.7. Let $A : H \to H$ be an $(r, \eta)$-strong monotone mapping and let $M : H \to H$ be an $(A, \eta)$-monotone mapping. Then the generalized resolvent operator $J^{A,\eta}_{M,\rho} : H \to H$ is defined by

$$J^{A,\eta}_{M,\rho}(u) = (A + \rho M)^{-1}(u), \quad \forall u \in H,$$

where $\rho > 0$ is a constant.

Notation ([12]). Let $A : H \to H$ be an $r$-strongly monotone mapping and let $M : H \to 2^H$ be an $A$-monotone mapping. Then the operator $(A + \rho M)^{-1}$ is single-valued.

Notation ([13], [14]). Let $\eta : H\times \to H$ be a single-valued mapping, $A : H \to H$ be $(r, \eta)$-strongly monotone mapping and $M : H \to 2^H$

be an $(A,\eta)$-monotone mapping. Then the mapping $(A+\rho M)^{-1}$ is single-valued.

In order to prove our main results, we also need the following lemmas.

LEMMA 2.8 ([16]). *Let $H$ be a real Hilbert space and let $\eta : H\times H \to H$ be a $\tau$-Lipschitz continuous nonlinear mapping. Let $A : H \to H$ be a $(r,\eta)$-strongly monotone and let $M : H \to 2^H$ be $(A,\eta)$-monotone. Then the generalized resolvent operator $J^{A,\eta}_{M,\rho} : H \to H$ is $\tau/(r-\rho m)$, that is,*

$$\|J^{A,\eta}_{M,\rho}(x) - J^{A,\eta}_{M,\rho}(y)\| \le \frac{\tau}{r-\rho m}\|x-y\|, \quad \forall x,y \in H.$$

LEMMA 2.9 ([16]). *Let $H$ be a real Hilbert space, let $A : H \to H$ be $(r,\eta)$-strongly monotone, and let $M : H \to 2^H$ be $(A,\eta)$-monotone. Let $\eta : H\times H \to H$ be a $\tau$-Lipschitz continuous nonlinear mapping. Then the following statements are mutually equivalent:*
*(i) An element $u \in H$ is a solution (1).*
*(ii) For $u \in H$, we have $u = J^{M,\eta}_{\rho,A}(A(u))$, where*

$$J^{M,\eta}_{\rho,A}(u) = (A+\rho M)^{-1}(u).$$

LEMMA 2.10. *Let $H$ be a real Hilbert space, let $A : H \to H$ be $(r,\eta)$-strongly monotone and $s$-Lipschitz continuous and let $M : H \to 2^H$ be $(A,\eta)$-monotone. Furthermore, let $\eta : H\times H \to H$ be $\tau$-Lipschitz continuous. Then*

$$\|J^{A,\eta}_{M,\rho}(A(u)) - J^{A,\eta}_{M,\rho}(A(v))\| \le \frac{s\tau}{r-\rho m}\|u-v\|^2.$$

*Consequently, we have*

$$\begin{aligned}&\langle (I-J^{A,\eta}_{M,\rho}A)(u) - (I-J^{A,\eta}_{M,\rho}A)(v), u-v\rangle \\ &\ge (1+\frac{s\tau}{r-\rho m})^{-2}(1-\frac{s\tau}{r-\rho m})\|(I-J^{A,\eta}_{M,\rho}A)(u) - (I-J^{A,\eta}_{M,\rho}A)(v)\|.\end{aligned}$$

*Proof.* Since $J^{A,\eta}_{M,\rho} : H \to H$ is $\tau/(r-\rho m)$ and $A$ is $s$-Lipschitz continuous, we can obtain the desired conclusion easily.

□

## 3. Main results

THEOREM 3.1. *Let $H$ be a real Hilbert space, let $A : H\times H$ be $(r,\eta)$-strongly monotone and $s$-Lipschitz continuous and let $M : H \to 2^H$ be $(A,\eta)$-monotone. Let $\eta : H\times H \to H$ be a $\tau$-Lipschitz continuous* *nonlinear mapping. For an arbitrarily chosen initial point $x_0$, suppose that the sequence $\{x_n\}$ is generated by the proximal point algorithm*

$$x_{n+1} \approx J_{M,\rho_n}^{A,\eta}(A(x_n))$$

*such that*

$$\|x_{n+1} - J_{M,\rho_n}^{A,\eta}(A(x_n))\| \le \epsilon_n,$$

*where $J_{M,\rho_n}^{A,\eta} = (A + \rho_n M)^{-1}$, $\epsilon_n \subset [0,\infty)$, $\rho_n \subset [0,\infty)$ are scalar sequences with $e_1 = \sum_{n=0}^{\infty} \epsilon_n < \infty$ and $\rho_n$ is bounded away from zero. Then the following conclusions hold:*

(i) *The sequence $\{x_n\}$ is bounded.*

(ii) $\lim_{n\to\infty} J_n^*(x_n) = 0$, *for $\rho_n < \frac{r-s\tau}{m}$ and $s\tau < r$.*

(iii) *The sequence $\{x_n\}$ converges weakly to a solution of (1.1).*

*Proof.* Suppose that $x^*$ is a zero of $M$. Put $J_n^* = I - J_{M,\rho_n}^{A,\eta}(A)$ for all $n \ge 0$. From Lemma 2.3, we have $J_n^*$ is $(1+\frac{s\tau}{r-\rho_n m})^2(1-\frac{s\tau}{r-\rho_n m})^{-1}$-firmly nonexpansive.

On the other hand, we have any solution to (1.1) is a fixed point of $J_{M,\rho_n}^{A,\eta}A$, and hence a zero of $J_n^*$. Observe that

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|x_{n+1} - J_{M,\rho_n}^{A,\eta}(A(x_n)) + J_{M,\rho_n}^{A,\eta}(A(x_n)) - x^*\| \\
&\le \epsilon_n + \|J_{M,\rho_n}^{A,\eta}(A(x_n)) - x^*\| \\
&\le \epsilon_n + \frac{s\tau}{r - \rho_n m}\|x_n - x^*\| \\
&\le e_1 + \|x_0 - x^*\|,
\end{aligned}$$

which yields that sequence $\{x_n\}$ is bounded. It follows from Lemma 2.3 that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|x_{n+1} - J_{M,\rho_n}^{A,\eta}(A(x_n)) - J_n^*(x_n) + x_n - x^*\|^2 \\
&= \|x_{n+1} - J_{M,\rho_n}^{A,\eta}(A(x_n)) - J_n^*(x_n)\|^2 + \|x_n - x^*\|^2 \\
&\quad + 2\langle x_{n+1} - J_{M,\rho_n}^{A,\eta}(A(x_n)), x_n - x^*\rangle - 2\langle J_n^*(x_n) - J_n^*(x^*), x_n - x^*\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \|x_{n+1} - J_{M,\rho_n}^{A,\eta}(A(x_n)) - J_n^*(x_n)\|^2 + \|x_n - x^*\|^2 \\
&\quad + 2\langle x_{n+1} - J_{M,\rho_n}^{A,\eta}(A(x_n)), x_n - x^*\rangle \\
&\quad - 2(1 + \frac{s\tau}{r - \rho_n m})^{-2}(1 - \frac{s\tau}{r - \rho_n m})\|J_n^*(x_n)\|^2 \\
&= \|x_{n+1} - J_{M,\rho_n}^{A,\eta}(A(x_n))\|^2 + \|J_n^*(x_n)\|^2 \\
&\quad - 2\langle x_{n+1} - J_{M,\rho_n}^{A,\eta}(A(x_n)), J_n^*(x_n)\rangle + \|x_n - x^*\|^2 \\
&\quad + 2\langle x_{n+1} - J_{M,\rho_n}^{A,\eta}(A(x_n)), x_n - x^*\rangle \\
&\quad - 2(1 + \frac{s\tau}{r - \rho_n m})^{-2}(1 - \frac{s\tau}{r - \rho_n m})\|J_n^*(x_n)\|^2 \\
&\le \epsilon_n^2 + \|J_n^*(x_n)\|^2 - 2\langle x_{n+1} - J_{M,\rho_n}^{A,\eta}(A(x_n)), J_n^*(x_n)\rangle + \|x_n - x^*\|^2 \\
&\quad + 2\langle x_{n+1} - J_{M,\rho_n}^{A,\eta}(A(x_n)), x_n - x^*\rangle \\
&\quad - 2(1 + \frac{s\tau}{r - \rho_n m})^{-2}(1 - \frac{s\tau}{r - \rho_n m})\|J_n^*(x_n)\|^2 \\
&\le \epsilon_n^2 + [1 + (1 + \frac{s\tau}{r - \rho_n m})^4(1 - \frac{s\tau}{r - \rho_n m})^{-2}]\|x_n - x^*\|^2 \\
&\quad + 2\langle x_{n+1} - J_{M,\rho_n}^{A,\eta}(A(x_n)), J_{M,\rho_n}^{A,\eta}(A(x_n)) - x^*\rangle \\
&\quad - 2(1 + \frac{s\tau}{r - \rho_n m})^{-2}(1 - \frac{s\tau}{r - \rho_n m})\|J_n^*(x_n)\|^2 \\
&\le \epsilon_n^2 + [1 + (1 + \frac{s\tau}{r - \rho_n m})^4(1 - \frac{s\tau}{r - \rho_n m})^{-2}]\|x_n - x^*\|^2 \\
&\quad + 2\epsilon_n \frac{s\tau}{r - \rho_n m}\|x_n - x^*\| \\
&\quad - 2(1 + \frac{s\tau}{r - \rho_n m})^{-2}(1 - \frac{s\tau}{r - \rho_n m})\|J_n^*(x_n)\|^2,
\end{aligned}
$$

where $\rho_n < \frac{r - s\tau}{m}$ and $s\tau < r$. It follows from the summability of the sequence $\{\epsilon_n\}$ that $e_2 = \sum_{n=0}^{\infty} \epsilon_n^2 < \infty$. Therefore, we have

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\le e_2 + [1 + (1 + \frac{s\tau}{r - \rho_n m})^4(1 - \frac{s\tau}{r - \rho_n m})^{-2}]\|x_0 - x^*\|^2 \\
&\quad + 2e_1 \frac{s\tau}{r - \rho_n m}\|x_0 - x^*\| \\
&\quad - 2(1 + \frac{s\tau}{r - \rho_n m})^{-2}(1 - \frac{s\tau}{r - \rho_n m}) \sum_{i=0}^{n} \|J_i^*(x_i)\|^2.
\end{aligned}
$$

We infer that $\sum_{i=0}^{n} \|J_i^*(x_i)\|^2 < \infty \Rightarrow \lim_{n\to\infty} J_n^*(x_n) = 0$. It follows that there exists a unique element $(\mu_n, \nu_n) \in M$ represented by $A(\mu_n) + \rho_n \nu_n = A(x_n)$ for all $n$. Observing $\mu_n = J_{M,\rho_n}^{A,\eta}(A(x_n))$ and $\lim_{n\to\infty} J_n^*(x_n) = 0$, we have

$$\lim_{n\to\infty} \|x_n - \mu_n\| = 0. \tag{3.1}$$

It further follows, since the $\rho_n$ is bounded away from zero, that

$$\lim_{n\to\infty} \frac{J_n^*(x_n)}{\rho_n} = \lim_{n\to\infty} \nu_n = 0.$$

Since the sequence $\{x_n\}$ is bounded and the space is a Hilbert space, we have that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\} \rightharpoonup x^*$. From (3.1), we have $\mu_{n_i}$ also converges weakly to $x^*$. Let some $(\mu, \nu) \in M$. It follows from $(A, \eta)$-monotonicity of $M$ that

$$\langle \mu - \mu_n, \eta(\nu, \nu_n)\rangle \geq (-m)\|\mu - \mu_n\|^2 \quad for\ all\ n \geq 0,$$

which yields that

$$\langle \mu - x^*, \eta(\nu, 0)\rangle \geq (-m)\|\mu - x^*\|^2 \quad for\ all\ n \geq 0.$$

Since $M$ is $(m, \eta)$-relaxed monotone and $(\mu, \nu)$ is arbitrary, we have $(x^*, 0) \in M$. This implies that $x^*$ is a solution of nonlinear variational inclusion (1.1).

Finally, since very Hilbert space is a *Opial's* space, we have that the whole sequence convergence weakly to $x^*$. This complete the proof. □

REMARK 3.2. Theorem 3.1 mainly improves Theorem 1 of Rockafellar [10] and also generalizes Theorem 3.2 of Verma [15] to the case of $(A, \eta)$-monotone mappings.

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