

THE INTEGRALS OF BANACH SPACE-VALUED FUNCTIONS

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ABSTRACT. In this paper, we define the ap-Henstock integral and the ap-Denjoy integral of Banach-valued functions, and we investigate some properties of these two integrals. In particular, we show that the ap-Henstock integral is equivalent to the ap-Denjoy integral.

1. Introduction and preliminaries

The ap-Denjoy integral of real valued functions was introduced in [13]. It is known [13] that the ap-Denjoy integral is equivalent to the ap-Henstock integral.

In this paper, we define the ap-Henstock integral and ap-Denjoy integral of Banach-valued functions, and we investigate the relationship of these two integrals.

Throughout this paper, X is a Banach space with dual X^* .

For a measurable set E of real numbers we denote by $|E|$ its Lebesgue measure. Let E be a measurable set and let c be a real number. The *density* of E at c is defined by

$$d_c E = \lim_{h \rightarrow 0^+} \frac{|E \cap (c - h, c + h)|}{2h}$$

provided the limit exists. The point c is called a *point of density* of E if $d_c E = 1$. The set E^d represents the set of all points $x \in E$ such that x is a point of density of E . A function $F : [a, b] \rightarrow X$ is said to be *approximately differentiable* at $c \in [a, b]$ if there exists a measurable set $E \subseteq [a, b]$ such that $c \in E^d$ and

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$$\lim_{\substack{x \rightarrow c \\ x \in E}} \frac{F(x) - F(c)}{x - c}$$

exists. The approximate derivative of F at c is denoted by $F'_{ap}(c)$.

An *approximate neighborhood* (of ap-nbd) of $x \in [a, b]$ is a measurable set $S_x \subseteq [a, b]$ containing x as a point of density. For every $x \in E \subseteq [a, b]$, choose an ap-nbd $S_x \subseteq [a, b]$ of x . Then we say that $S = \{S_x : x \in E\}$ is a *choice* on E . A tagged interval $(x, [c, d])$ is said to be *subordinate* to the choice $S = \{S_x\}$ if $c, d \in S_x$. Let $\mathcal{P} = \{(x_i, [c_i, d_i]) : 1 \leq i \leq n\}$ be a finite collection of non-overlapping tagged intervals. If $(x_i, [c_i, d_i])$ is subordinate to a choice S for each i , then we say that $\mathcal{P}$ is subordinate to S . Let $E \subseteq [a, b]$. If $\mathcal{P}$ is subordinate to S and each $x_i \in E$, then $\mathcal{P}$ is called *E-subordinate* to S . If $\mathcal{P}$ is subordinate to S and $[a, b] = \bigcup_{i=1}^n [c_i, d_i]$, then we say that $\mathcal{P}$ is a *tagged partition* of $[a, b]$ that is subordinate to S .

2. The ap-Henstock integral of Banach-valued functions

We introduce the ap-Henstock integral of Banach-valued functions.

DEFINITION 2.1. A function $f : [a, b] \rightarrow X$ is *ap-Henstock integrable* on $[a, b]$ if there exists a vector $A \in X$ with the following property: for each $\epsilon > 0$ there exists a choice S on $[a, b]$ such that $\|f(\mathcal{P}) - A\| < \epsilon$ whenever $\mathcal{P}$ is a tagged partition of $[a, b]$ that is subordinate to S , where $f(\mathcal{P}) = (\mathcal{P}) \sum f(x)|I|$. The vector A is called the ap-Henstock integral of f on $[a, b]$ and is denoted by $(AH) \int_a^b f$.

If f is ap-Henstock integrable on $[a, b]$, then f is also ap-Henstock integrable on any subinterval of $[a, b]$. Hence, an interval function F can be defined by $F(I) = (AH) \int_I f$. The function F is called the primitive of f .

It is easy to show the following theorem.

THEOREM 2.1. *Let f and g be functions mapping $[a, b]$ into X .*

(a) *If f is ap-Henstock integrable on each $[a, c]$ and $[c, b]$, then f is ap-Henstock integrable on $[a, b]$ and*

$$(AH) \int_a^b f = (AH) \int_a^c f + (AH) \int_c^b f$$

(b) If f and g are ap-Henstock integrable on $[a, b]$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ is ap-Henstock integrable on $[a, b]$ and

$$(AH) \int_a^b (\alpha f + \beta g) = \alpha (AH) \int_a^b f + \beta (AH) \int_a^b g$$

THEOREM 2.2. Let $f : [a, b] \rightarrow X$ be ap-Henstock integrable on $[a, b]$. Then for each $x^* \in X^*$ the function $x^* f$ is ap-Henstock integrable on $[a, b]$ and

$$x^* (AH) \int_a^b f = (AH) \int_a^b x^* f$$

Proof. Since f is ap-Henstock integrable on $[a, b]$, for every $\epsilon > 0$ there exists a choice S on $[a, b]$ such that for any partition $\mathcal{P} = \{(x, I)\}$ that is subordinate to S we have

$$\|f(\mathcal{P}) - (AH) \int_a^b f\| < \epsilon$$

For any $x^* \in X^*$, we have

$$|x^* f(\mathcal{P}) - x^* (AH) \int_a^b f| \leq \|x^*\| \|f(\mathcal{P}) - (AH) \int_a^b f\| < \|x^*\| \epsilon.$$

Hence, $x^* f$ is ap-Henstock integrable on $[a, b]$ and

$$(AH) \int_a^b x^* f = x^* (AH) \int_a^b f.$$

□

THEOREM 2.3. Let $f : [a, b] \rightarrow X$ be a function. If $f = 0$ almost everywhere on $[a, b]$, then f is ap-Henstock integrable on $[a, b]$ and $(AH) \int_a^b f = 0$.

Proof. Let $E = \{x \in [a, b] : \|f(x)\| \neq 0\}$ and for each positive integer n , let $E_n = \{x \in E : n - 1 \leq \|f(x)\| < n\}$. Let $\epsilon > 0$. For each n , choose an open set O_n such that $E_n \subseteq O_n$ and $|O_n| < \epsilon/n2^n$.

Define a choice $S = \{S_x : x \in [a, b]\}$ by

$$S_x = \begin{cases} [a, b] & \text{if } x \in [a, b] - E, \\ O_n & \text{if } x \in E_n. \end{cases}$$

Suppose that $\mathcal{P}$ is a tagged partition of $[a, b]$ that is subordinate to S . For each n , let $\mathcal{P}_n$ be the subset of $\mathcal{P}$ that has tag in E_n . We have

$$\|f(\mathcal{P})\| \leq \sum_{n=1}^{\infty} \|f(\mathcal{P}_n)\| < \sum_{n=1}^{\infty} n |O_n| < \sum_{n=1}^{\infty} \epsilon/2^n = \epsilon.$$

Hence, f is ap-Henstock integrable on $[a, b]$ and $(AH)\int_a^b f = 0$. $\square$

THEOREM 2.4. *Let $f : [a, b] \rightarrow X$ be ap-Henstock integrable on $[a, b]$. Then*

- (a) *the function f is weakly measurable.*
- (b) *If $g = f$ almost everywhere on $[a, b]$, then g is ap-Henstock integrable on $[a, b]$ and*

$$(AH)\int_a^b f = (AH)\int_a^b g.$$

Proof. (a) Since x^*f is ap-Henstock integrable for each $x^* \in X^*$, x^*f is measurable. Hence f is weakly measurable.

(b) Since $f - g = 0$ almost everywhere on $[a, b]$, $f - g$ is ap-Henstock integrable on $[a, b]$ and $(AH)\int_a^b (f - g) = 0$ by Theorem 2.3. Hence, $g = f - (f - g)$ is ap-Henstock integrable on $[a, b]$ and

$$(AH)\int_a^b f - (AH)\int_a^b g = (AH)\int_a^b (f - g) = 0$$

$\square$

DEFINITION 2.2. A function $F : [a, b] \rightarrow X$ is AC_s on a measurable set $E \subseteq [a, b]$ if for each $\epsilon > 0$ there exist a positive number δ and a choice S on E such that $\|(\mathcal{P}) \sum F(I)\| < \epsilon$ for every finite collection $\mathcal{P}$ of non-overlapping tagged intervals that is subordinate to S and satisfies $(\mathcal{P}) \sum |I| < \delta$. The function F is ACG_s on E if E can be expressed as a countable union of measurable sets on each of which F is AC_s .

LEMMA 2.5. *Suppose that $f : [a, b] \rightarrow X$ and let $E \subseteq [a, b]$. If $|E| = 0$, then for each $\epsilon > 0$ there exists a choice S on E such that $\|f\|(\mathcal{P}) < \epsilon$ whenever $\mathcal{P}$ is E -subordinate to S , where $\|f\|(\mathcal{P}) = (\mathcal{P}) \sum \|f(x)\||I|$.*

Proof. For each positive integer n , let $E_n = \{x \in E : n-1 \leq \|f(x)\| < n\}$. Then $E = \bigcup_{n=1}^{\infty} E_n$. Let $\epsilon > 0$. For each n , let $S^n = \{S_x^n : x \in E_n\}$ be a choice on E_n and choose an open set O_n such that $E_n \subseteq O_n$ and $|O_n| < \epsilon/n2^n$. For each $x \in E_n$, let $S_x = S_x^n \cap O_n$. Then $S = \{S_x : x \in E\}$ is a choice on E .

Suppose that $\mathcal{P}$ is E -subordinate to S . Let $\mathcal{P}_n$ be the subset of $\mathcal{P}$ that tags in E_n . Then we have

$$\|f\|(\mathcal{P}) = \sum_{n=1}^{\infty} \|f\|(\mathcal{P}_n) < \sum_{n=1}^{\infty} n|O_n| < \sum_{n=1}^{\infty} \epsilon/2^n = \epsilon.$$

$\square$

LEMMA 2.6. Suppose that $F : [a, b] \rightarrow X$ is ACG_s on $[a, b]$ and let E be a subset of $[a, b]$. If $|E| = 0$, then for each $\epsilon > 0$ there exists a choice S of E such that $\|(\mathcal{P}) \sum F(I)\| < \epsilon$ for each $\mathcal{P}$ that is subordinate to S .

Proof. Suppose that $E \subseteq [a, b]$ is a measurable set of measure zero. Let $E = \bigcup_{n=1}^{\infty} E_n$, where $\{E_n\}$ is a sequence of disjoint measurable sets and F is AC_s on each E_n . Let $\epsilon > 0$. For each positive integer n , there exist a choice $S^n = \{S_x^n : x \in E_n\}$ on E_n and a positive number δ_n such that $\|(\mathcal{P}) \sum F(I)\| < \epsilon/2^n$ whenever $\mathcal{P}$ is E_n -subordinate to S^n and $(\mathcal{P}) \sum |I| < \delta_n$. For each positive integer n , choose an open set O_n such that $E_n \subseteq O_n$ and $|O_n| < \delta_n$. Let $S_x = S_x^n \cap O_n$ for each $x \in E_n$. Then $S = \{S_x : x \in E\}$ is a choice on E . Suppose that $\mathcal{P}$ is E -subordinate to S . Let $\mathcal{P}_n$ be a subset of $\mathcal{P}$ that has tags in E_n and note that $(\mathcal{P}_n) \sum |I| < |O_n| < \delta_n$. Hence, we have

$$\|(\mathcal{P}) \sum F(I)\| \leq \sum_{n=1}^{\infty} \|(\mathcal{P}_n) \sum F(I)\| < \sum_{n=1}^{\infty} \epsilon/2^n = \epsilon.$$

□

THEOREM 2.7. If there exists an ACG_s function F on $[a, b]$ such that $F'_{ap} = f$ almost everywhere on $[a, b]$, then the function f is *ap-Henstock integrable* on $[a, b]$.

Proof. Suppose that there exists an ACG_s function F on $[a, b]$ such that $F'_{ap} = f$ almost everywhere on $[a, b]$. Let

$$E = \{x \in [a, b] : F'_{ap}(x) \neq f(x)\}.$$

Then $|E| = 0$. Let $D = [a, b] - E$ and let $\epsilon > 0$. For each $x \in D$, there exists a measurable set $D_x \subseteq [a, b]$ such that $x \in D_x^d$ and

$$F'_{ap}(x) = \lim_{\substack{y \rightarrow x \\ y \in D_x}} \frac{F(y) - F(x)}{y - x}.$$

Hence, there exists $\delta_x > 0$ such that for every $y \in D_x \cap (x - \delta_x, x + \delta_x) = S_x$

$$\|F(y) - F(x) - F'_{ap}(x)(y - x)\| \leq \epsilon|y - x|.$$

If $(x, [u, v])$ is a tagged interval with $u, v \in S_x$, then

$$\begin{aligned} & \|F(v) - F(u) - F'_{ap}(x)(v - u)\| \\ & \leq \|F(v) - F(x) - F'_{ap}(x)(v - x)\| + \|F(x) - F(u) - F'_{ap}(x)(x - u)\| \\ & < \epsilon(v - x) + \epsilon(x - u) = \epsilon(v - u). \end{aligned}$$

Hence, there exists a choice $S' = \{S_x : x \in D\}$ on D such that $\|f(\mathcal{P}) - F(\mathcal{P})\| < \epsilon(\mathcal{P}) \sum |I|$ whenever $\mathcal{P}$ is a collection of tagged intervals that is subordinate to S' .

By Lemmas 2.5 and 2.6, there exists a choice S'' on E such that $\|f(\mathcal{P})\| < \epsilon$ and $\|F(\mathcal{P})\| < \epsilon$ whenever $\mathcal{P}$ is subordinate to S'' . let $S = S' \cup S''$. Then S is a choice on $[a, b]$.

Suppose that $\mathcal{P}$ is a tagged partition of $[a, b]$ that is subordinate to S . Let $\mathcal{P}_E$ be the subset of $\mathcal{P}$ that has tags in E and let $\mathcal{P}_D = \mathcal{P} - \mathcal{P}_E$. Then we have

$$\begin{aligned} \|f(\mathcal{P}) - F(\mathcal{P})\| &\leq \|f(\mathcal{P}_D) - F(\mathcal{P}_D)\| + \|f(\mathcal{P}_E)\| + \|F(\mathcal{P}_E)\| \\ &< \epsilon(b - a + 2). \end{aligned}$$

Hence, f is ap-Henstock integrable on $[a, b]$. $\square$

The ap-Henstock integral has the following gemetric property.

THEOREM 2.8. *If $f : [a, b] \rightarrow X$ is ap-Henstock integrable on $[a, b]$, then for every integral $[c, d] \subseteq [a, b]$,*

$$\frac{1}{d - c} (AH) \int_c^d f \in \overline{\text{co}} f([c, d]),$$

where $\overline{\text{co}} f([c, d])$ is the closed convex hull of $f([c, d])$.

Proof. Suppose that there exists an interval $[c, d]$ such that

$$\frac{1}{d - c} (AH) \int_c^d f \notin \overline{\text{co}} f([c, d]).$$

By the Hahn-Banach Theorem, we can select $x^* \in X^*$ and a real number α such that

$$x^* \left(\frac{1}{d - c} (AH) \int_c^d f \right) < \alpha \leq x^* f(t)$$

for all $t \in [c, d]$. Then

$$\frac{1}{d - c} (AH) \int_c^d x^* f < \alpha \leq x^* f(t).$$

Integrating over $[c, d]$, we have

$$(AH) \int_c^d x^* f < \alpha(d - c) \leq (AH) \int_c^d x^* f.$$

This is a contradiction. $\square$

3. The ap-Denjoy integral of Banach-valued functions.

We introduce the notion of the approximate Lusin function. This function is used to define the ap-Denjoy integral.

For a function $F : [a, b] \rightarrow X$, F can be treated as a function of intervals by defining $F([c, d]) = F(d) - F(c)$.

DEFINITION 3.1. Let $F : [a, b] \rightarrow X$ be a function. The function F is an *approximate Lusin function* (or F is an *AL function*) on $[a, b]$ if for every measurable set $E \subseteq [a, b]$ of measure zero and for every $\epsilon > 0$ there exists a choice S on E such that $\|(\mathcal{P}) \sum F(I)\| < \epsilon$ for every finite collection $\mathcal{P}$ of non-overlapping tagged intervals that is E -subordinate to S .

From Lemma 2.6, we get the following theorem

THEOREM 3.1. If $F : [a, b] \rightarrow X$ is ACG_s on $[a, b]$, then F is an *AL function* on $[a, b]$.

DEFINITION 3.2. A function $f : [a, b] \rightarrow X$ is *ap-Denjoy integrable* on $[a, b]$ if there exists an *AL function* F on $[a, b]$ such that F is approximately differentiable almost everywhere on $[a, b]$ and $F'_{ap} = f$ almost everywhere on $[a, b]$. The function f is ap-Denjoy integrable on a measurable set $E \subseteq [a, b]$ if $f\chi_E$ is ap-Denjoy integrable on $[a, b]$.

If we add the condition $F(a) = 0$, then the function F is unique. We will denote this function $F(x)$ by $(AD)\int_a^x f$.

It is easy to show that if $f : [a, b] \rightarrow X$ is ap-Denjoy integrable on $[a, b]$, then f is ap-Denjoy integrable on every subinterval of $[a, b]$. This gives rise to an interval function F such that $F(I) = (AD)\int_I f$ for every subinterval $I \subseteq [a, b]$. The function F is called the primitive of f .

From the definition of the ap-Denjoy integral, we get the following theorem.

THEOREM 3.2. Let $f : [a, b] \rightarrow X$ be ap-Denjoy integrable on $[a, b]$ and let $F(x) = (AD)\int_a^x f$ for each $x \in [a, b]$. Then the function F is approximately differentiable almost everywhere on $[a, b]$ and $F'_{ap} = f$ almost everywhere on $[a, b]$.

THEOREM 3.3. Let $f : [a, b] \rightarrow X$ and let $c \in (a, b)$.

(a) If f is ap-Denjoy integrable on $[a, b]$, then f is ap-Denjoy integrable on every subinterval of $[a, b]$.

(b) If f is ap-Denjoy integrable on each of the intervals $[a, c]$ and $[c, b]$, then f is ap-Denjoy integrable on $[a, b]$ and

$$(AD)\int_a^b f = (AD)\int_a^c f + (AD)\int_c^b f.$$

Proof. (a) Let $[c, d]$ be any subinterval on $[a, b]$. Let $F(x) = (AD)\int_a^x f$ for each $x \in [a, b]$. Since F is an AL function on $[a, b]$ and $F'_{ap} = f$ almost everywhere on $[a, b]$, F is an AL function on $[c, d]$ and $F'_{ap} = f$ almost everywhere on $[c, d]$. Hence, f is ap-Denjoy integrable on $[c, d]$.

(b) Since f is ap-Denjoy integrable on each of intervals $[a, c]$ and $[c, b]$, there exist AL functions F and G such that $F'_{ap} = f$ almost everywhere on $[a, c]$ and $G'_{ap} = f$ almost everywhere on $[c, b]$ respectively. Define $H : [a, b] \rightarrow X$ by

$$H(x) = \begin{cases} F(x), & \text{if } x \in [a, c]; \\ F(c) + G(x), & \text{if } x \in (c, b]. \end{cases}$$

Then H is an AL function on $[a, b]$ and $H'_{ap} = f$ almost everywhere on $[a, b]$. Hence f is ap-Denjoy integrable on $[a, b]$ and $H(b) = F(c) + G(b)$, i.e.,

$$(AD)\int_a^b f = (AD)\int_a^c f + (AD)\int_c^b f$$

□

We can easily get the following theorem.

THEOREM 3.4. Suppose that f and g are ap-Denjoy integrable on $[a, b]$. Then

(a) kf is ap-Denjoy integrable on $[a, b]$ and $(AD)\int_a^b kf = k(AD)\int_a^b f$ for each $k \in \mathbb{R}$,

(b) $f + g$ is ap-Denjoy integrable on $[a, b]$ and $(AD)\int_a^b (f + g) = (AD)\int_a^b f + (AD)\int_a^b g$.

THEOREM 3.5. A function $f : [a, b] \rightarrow X$ is ap-Denjoy integrable on $[a, b]$ if and only if there exists an ACG_s function F on $[a, b]$ such that $F'_{ap} = f$ almost everywhere on $[a, b]$.

Proof. Suppose that there exists an ACG_s function F on $[a, b]$ such that $F'_{ap} = f$ almost everywhere on $[a, b]$. Then F is an AL function by Theorem 3.1. Hence, f is ap-Denjoy integrable on $[a, b]$.

Conversely, suppose that f is ap-Denjoy integrable on $[a, b]$ and let $F(x) = (AD)\int_a^x f$ for each $x \in [a, b]$. Then F is an AL function such

that $F'_{ap} = f$ almost everywhere on $[a, b]$. Let $E = \{x \in [a, b] : F'_{ap}(x) \neq f(x)\}$. Then $|E| = 0$. Since F is an AL function, F is AC_s on E . For each positive integer n , let

$$E_n = \{x \in [a, b] - E : n - 1 \leq \|f(x)\| < n\}.$$

Fix n and let $\epsilon > 0$. Since F is approximately differentiable for each $x \in E_n$, there exist a measurable set A_x containing x as a point of density and a positive number δ_x such that

$$\left\| \frac{F(y) - F(x)}{y - x} - f(x) \right\| < \epsilon$$

i.e.,

$$\|F(y) - F(x) - f(x)(y - x)\| < \epsilon|y - x|,$$

if $y \in A_x \cap (x - \delta_x, x + \delta_x)$. For each $x \in E_n$, let

$$S_x = A_x \cap (x - \delta_x, x + \delta_x)$$

Then $S = \{S_x : x \in E_n\}$ is a choice E_n . Suppose that $\mathcal{P}$ is a finite collection of non-overlapping tagged intervals that is E_n -subordinate to S and satisfies $\mu(\mathcal{P}) < \frac{\epsilon}{n}$. Then since $\|F(\mathcal{P}) - f(\mathcal{P})\| < \epsilon\mu(\mathcal{P})$, we have

$$\begin{aligned} \|F(\mathcal{P})\| &\leq \|F(\mathcal{P}) - f(\mathcal{P})\| + \|f(\mathcal{P})\| \\ &< \epsilon\mu(\mathcal{P}) + n\mu(\mathcal{P}) \\ &< (b - a + 1)\epsilon \end{aligned}$$

Hence, F is AC_s on E_n . Since $[a, b] = [\cup_{n=1}^{\infty} E_n] \cup E$, F is ACG_s on $[a, b]$. $\square$

THEOREM 3.6. *Let $f : [a, b] \rightarrow X$ be ap-Denjoy integrable on $[a, b]$ and let $F(x) = (AD)\int_a^x f$ for each $x \in [a, b]$. Then F is approximately continuous on $[a, b]$.*

Proof. From the definition of the ap-Denjoy integral, F is approximately differentiable almost everywhere on $[a, b]$. Let E be the set of all non-differentiable points in $[a, b]$. Then E is a measurable set of measure zero. Since F is approximately continuous on $[a, b] - E$, it is sufficient to show that F is approximately continuous on E . Let $c \in E$ and let $\epsilon > 0$. Since F is an AL function, there exists a choice $S = \{S_x : x \in E\}$ such that $\|(\mathcal{P}) \sum F(I)\| < \epsilon$ for every finite collection $\mathcal{P}$ of non-overlapping tagged intervals that is E -subordinate to S . If $x \in S_c \cap (c - \eta, c + \eta)$ for some $\eta > 0$, then the tagged interval $(c, [c, x])$ (or $(c, [x, c])$) is E -subordinate to S . Hence, $\|F(x) - F(c)\| = \|F([c, x])\| < \epsilon$. This shows that F is approximately continuous on E . $\square$

From Theorem 2.7 and 3.5, we can get the following result.

THEOREM 3.7. *If a function $f : [a, b] \rightarrow X$ is ap-Denjoy integrable on $[a, b]$, then f is ap-Henstock integrable on $[a, b]$.*

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