

CIRCULAR UNITS IN A BICYCLIC FUNCTION FIELD

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ABSTRACT. For a real subextension of some cyclotomic function field with a non-cyclic Galois group order l^2 , l being a prime different from the characteristic of function field, we compute the index of the Sinnott group of circular units.

1. Introduction

The group E of units of an abelian number field K , though finitely generated, is very difficult to compute. However, it contains the explicitly described group C of circular units, which has a finite index in E . In 1980 Sinnott defined his generalisation of a group of circular units to an abelian number field and found a general formula for the index ([4]):

$$[E : C] = h_K 2^{[K:\mathbb{Q}]-1} \frac{\prod_{p|f_K} [K_{p^e} : \mathbb{Q}]}{[K : \mathbb{Q}]} (R : U)$$

where $f_K = \prod p^e$ is the decomposition of the conductor f_K of K into prime factors, $K_{p^e} = K \cap \mathbb{Q}_{p^e}$, and $(R : U)$ is the Sinnott index of the field K . The definition of $(R : U)$ is quite complicated, it is only known to very special cases. In [2], Kraemer computed the index $[E : C]$ explicitly and easily got the value of the Sinnott's index $(R : U)$ from the Sinnott's formula.

From [1], we see that

$$[\mathcal{O}_F^* : C_F] = \left(\frac{q-1}{\delta_F}\right)^{[F^+:k]-1} Q_0 h(\mathcal{O}_{F^+}) \frac{\prod_{i=1}^s [F_{\mathfrak{p}_i^{e_i}} : F_{\mathfrak{e}}]}{[F : F_{\mathfrak{e}}][K_{\mathfrak{e}} : F_{\mathfrak{e}}]} \frac{(e^+ R : e^+ U)}{d(F)}.$$

for a subfield F of some cyclotomic function field over a global function field k . Suppose l is a prime not equal to $\text{char}(k)$ and F a real abelian extension of degree l^2 over k with a non-cyclic Galois group and F is

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contained in some cyclotomic function field over $\mathbb{F}_q(T)$. Then we have $\delta_F = 1, Q_0 = 1$ ([1, Lemma 2.2]), $d(F) = 1$. Therefore we have

$$(1.1) \quad [E : C] = (q - 1)^{l^2 - 1} h(\mathcal{O}_F) l^{v-2} (R : U),$$

where v is the number of subfields of degree l of F , which conductor is a power of irreducible polynomials. As in the number field case, the Sinnott index $(R : U)$ is difficult to compute and known only some special cases.

In this paper we will consider the case of a real subextension F of a cyclotomic function field with a non-cyclic Galois group of order l^2 , where l is a prime different from the characteristic of k . First, we find a basis of C/B , where B is the product of the groups of circular units of all proper subfields of F . Next, we directly compute $[E : B]$, which gives us the index $[E : C]$. From the index-formula (1.1), we easily compute the Sinnott's index $(R : U)$.

Notation

$A = \mathbb{F}_q[T], k = \mathbb{F}_q(T)$

F : a real abelian extension of degree l^2 over k with a non-cyclic Galois group. Here l is a prime, not equal to $\text{char}(k)$. We assume that F is contained in some cyclotomic function field.

$G = \text{Gal}(F/k)$, the Galois group of F over k .

R_F : the regulator of F

$h(F)$ (resp. $h(\mathcal{O}_F)$) : the divisor (resp. ideal) class number of F

F_i, f_i : subfields of F of degree l and their conductors ($0 \leq i \leq l$)

$G_i = \text{Gal}(F/F_i)$

λ_N : the primitive N -torsion point. Here $N \in \mathbb{A}$

$K_{P_i^{e_i}} = K(\lambda_{P_i^{e_i}})$ $P_i^{e_i}$ -th cyclotomic function field

X_L : the group of Dirichlet characters corresponding to a field L

f_χ : the conductor of a character χ

$I_P \subseteq G$: the inertia group of P

E, D, C : the group of units, circular numbers and circular units of F

2. Circular units

Since $(\mathbb{A}/P)^*$ is cyclic for any irreducible polynomials P and G is not cyclic, the conductor f of F can not be a power of an irreducible polynomial. Therefore f is divisible by at least two different monic irreducible polynomials. The group $G = \text{Gal}(F/k) \cong \mathbb{F}_l \times F_l$ has exactly

$l + 1$ subgroups G_i of order l . Let $F_i \subset F$ be the subfield of degree l corresponding to G_i . Let δ_i be a fixed generator of $\text{Gal}(F_i/k)$.

LEMMA 2.1. *The conductor $f = \text{cond}(F)$ of F has no square factors.*

Proof. If $P^2|f$, then $P^2|f_\chi$ for some $\chi \in X_F$. If $\chi = \chi_P \chi_{P_1} \cdots \chi_{P_t}$ is a decomposition of χ , then $P^2|f_{\chi_P}$. Then $p = \text{char}(k) || \langle \chi_P \rangle$. Therefore p must divide the ramification index of p in F ([5, Theorem 3.5].) Hence $p|[F : k] = l^2$, which is a contradiction. $\square$

From the same argument in [2], we have

PROPOSITION 2.2. *Let $P_1, \dots, P_s$ be all the monic irreducible polynomials ramifying in F . If $Q_i = \{j | P_j \nmid f_i, 1 \leq j \leq s\}$ is the index set for the monic irreducibles unramified in F_i , the the sets $Q_0, \dots, Q_l$ are mutually disjoint proper subsets of the set $Q = \{1, \dots, s\}$ and we have*

$$Q = \bigcup_{i=0}^l Q_i$$

The group D of cyclotomic numbers of F is the group generated by $\mathbb{F}_q^*, \varepsilon = N_{K_f/F}(\lambda_f), \varepsilon_i = N_{K_{f_i}/F_i}(\lambda_{f_i})$.

PROPOSITION 2.3. *$C/\mathbb{F}_q^*$ is a G -module generated by*

$$\eta = N_{K_f/F}(\lambda_f),$$

$$\eta_i = \begin{cases} N_{K_{f_i}/F_i}(\lambda_{f_i}) & \text{for } f_i \text{ composite,} \\ N_{K_{f_i}/F_i}(\lambda_{f_i})^{(1-\delta_i)} & \text{for } f_i \text{ irreducible.} \end{cases}$$

Notation : For $P_j \nmid f_i$,

$$(\delta_i|_{F_i})^{k_{ij}} = \text{Frob}^{-1}(P_j, F_i),$$

$k_{ij} \in \mathbb{F}_l$

From the almost same argument in [2],

PROPOSITION 2.4. *We have*

$$\eta^l = \prod_{i=0}^l \varepsilon_i^{\prod_{j \in Q_i} (1-\delta^{k_{ij}})}.$$

COROLLARY 2.5.

$$\eta^l = \prod_{i=0}^l \eta_i^{\beta_i},$$

where $\beta_i \in \mathbb{Z}[\delta_i]$ are defined as follows:

$$\beta_i = \begin{cases} \prod_{j \in Q_i} (1 - \delta_i^{k_{ij}}) & \text{if } f_i \text{ composite,} \\ \prod_{j \in Q_i} (1 - \delta_i^{k_{ij}}) / (1 - \delta_i) & \text{if } f_i \text{ irreducible,} \end{cases}$$

Let B be the subgroup of $C/\mathbb{F}_q^*$ as a G -module generated by $\{\eta_i\}$. We see that $B = \prod_{i=1}^{l+1} C_i/\mathbb{F}_q^*$, where C_i is the group of circular units of F_i .

PROPOSITION 2.6. *The set $Z = \{\eta_i^{\delta_i^j} : 0 \leq i \leq l, 0 \leq j \leq l-2\}$ is a $\mathbb{Z}$ -basis of B .*

Definition : For $0 \leq i \leq l$ we define:

$$a_i = \begin{cases} -\infty & \text{if there is a } j \in Q_i \text{ such that } k_{ij} = 0, \\ l-1-|Q_i| & \text{if there is no such } j \in Q_i \text{ and } f_i \text{ is composite,} \\ l-|Q_i| & \text{if there is no such } j \in Q_i \text{ and } f_i \text{ is irreducible,} \end{cases}$$

We may assume that G_i, K_i are chosen such that

$$a_0 \geq a_1 \geq \dots \geq a_l.$$

Let σ (resp. τ) be a fixed generator of G_0 (resp. G_1). For $2 \leq i \leq l$, let $\tau\sigma^{l-n_i}$ be a fixed generator of G_i , which defines $n_i \in \mathbb{F}_l^*$ in a unique way. If we define $n_1 = 0$, $\tau|_{F_i} = \sigma^{n_i}|_{F_i}$ for $1 \leq i \leq l$. We may assume that $\delta_0 = \tau, \delta_i = \sigma$ for $1 \leq i \leq l$.

Notation : Let k (resp. q) be the smallest i such that $a_i \leq 0$ (resp. $a_i = -\infty$) (if no such i exists then k (resp. q) equals $l+1$).

Notation : Put $p_{ij} = \eta^{(1-\tau)^i(1-\sigma)^j}$.

$$S_0 = \{p_{ij} : j = 0, 0 \leq i < a_0\}, R_{m,n} = \{p_{ij} : i = m, 0 < j < n\},$$

$$U = \begin{cases} \emptyset & \text{for } k = 0, \\ S_0 & \text{for } k = 1, \\ S_0 \cup R_{0,a_1} \cup R_{1,a_2-1} \cup \dots \cup R_{b-1,a_b-(b-1)} & \text{for } k \geq 2, \end{cases}$$

where $b = 1$ for $k = 2$, for $k > 2$, it is the greatest index $i < k$ such that $a_i > i$.

THEOREM 2.7. *The set U is a basis of C/B .*

Proof. The proof is the almost same as in the number field case ([2, Section 4]). $\square$

COROLLARY 2.8.

$$[C : B] = \begin{cases} 1 & \text{for } k = 0, \\ l^{a_0} & \text{for } k = 1, \\ l^{a_0 + (a_1 - 1) + \dots + (a_b - b)} & \text{for } k \geq 2, \end{cases}$$

where b is defined as above.

3. Computation of the index $[\mathbf{E}:\mathbf{C}]$

Let ∞_F be a fixed place of F lying above ∞ . From [6, Proposition 4.1], we have

LEMMA 3.1. *Let $\chi \in X_{F_i}$. Then we have*

$$\sum_{g \in G} \chi(g) \text{ord}_{\infty_F}(\eta_i^g) = \begin{cases} l(q-1)L_k(0, \chi) & \text{if } f_i \text{ composite,} \\ l(q-1)(1 - \chi(\delta_i)^{-1})L_k(0, \chi) & \text{if } f_i \text{ irreducible.} \end{cases}$$

First we compute $[E : B]$.

PROPOSITION 3.2. *Let v be the number of subfields of F having an irreducible conductor. Then we have*

$$[E : B] = l^{v-2 + \frac{l(l-1)}{2}} (q-1)^{l^2-1} h(\mathcal{O}_F).$$

Proof. The proof is similar to the number field case([2, Proposition 5.1]). The regulator of the basis Z of B is

$$R_B = |\det(\text{ord}_{\infty_F}(\eta_i^{\delta_j g}))|_{0 \leq i \leq l, 0 \leq j \leq l-2, 1 \neq g \in G}.$$

We enlarge the matrix to

$$M = \begin{pmatrix} 1 & \dots & 1 & \dots \\ \vdots & & \vdots & \\ \text{ord}_{\infty_F}(\eta_i^{\delta_j}) & \dots & \text{ord}_{\infty_F}(\eta_i^{\delta_j g}) & \dots \\ \vdots & & \vdots & \end{pmatrix}$$

Put $Z = (\chi(g))_{g \in G, \chi \in X}$. Note that $|\det(M)| = l^2 R_B, |\det(Z)| = l^{l^2}$. We assume that the first row (resp. column) corresponds to the trivial automorphism (resp. character). The other rows (resp. columns) will be indexed with ordered pairs (i, j) (resp. (m, n)) with $0 \leq i, m \leq l$ and $1 \leq j, n \leq l-1$ and arranged in an increasing lexicographical order. The row (i, j) (resp. column (m, n)) corresponds to the automorphism g_i^j (resp. character χ_m^n), where g_i (resp. χ_m) denotes a fixed generator of G_i (resp. X_{K_m}).

Now we evaluate MZ :

1. In the first row we get the numbers $a_\chi = \sum_{g \in G} \chi(g)$, so we have $(l^2, 0, \dots, 0)$. 2. At $(i, j), (m, n)$, with $i \neq m$ we get

$$a_{(i,j),(m,n)} = 0.$$

3. At $(i, j), (i, n)$ we get

$$a_{(i,j),(m,n)} = \sum_{g \in G} \chi_{i,n}(g) \text{ord}_{\infty_F}(\eta_i^{\delta_i^j g}) = \chi_{i,n}(\delta_i)^{-j} c_{in},$$

where $c_{in} = \sum_{g \in G} \chi_{i,n}(g) \text{ord}_{\infty_F}(\eta_i^g)$

Thus we have

$$\begin{aligned} |\det(MZ)| &= l^2 \prod_{i=0}^l |\det(a_{(i,j),(i,n)})_{0 \leq j \leq l-2, 1 \leq n \leq l-1}| \\ &= l^2 \prod_{i=0}^l \left(\prod_{n=1}^{l-1} |c_{in}| \right) \cdot |\det(\chi_{i,n}(\delta_i)^{-j})_{0 \leq j \leq l-2, 1 \leq n \leq l-1}| \end{aligned}$$

By Lemma 3.1, we have

$$\prod_{n=1}^{l-1} |c_{in}| = \begin{cases} l^{l-1} (q-1)^{l-1} \prod_{n=1}^{l-1} |L_k(0, \chi_{i,n})| & (f_i \text{ comp.}), \\ l^{l-1} (q-1)^{l-1} \prod_{n=1}^{l-1} |1 - \chi_{i,n}(\delta_i^{-1})| |L_k(0, \chi_{i,n})| & (f_i \text{ irred.}), \end{cases}$$

and

$$\prod_{n=1}^{l-1} |1 - \chi_{i,n}(\delta_i^{-1})| = l$$

Therefore we have

$$\begin{aligned} |\det(MZ)| &= l^2 l^{(l+1)(l-1)} (q-1)^{(l+1)(l-1)} l^v \prod_{i=0}^l \prod_{n=1}^{l-1} |L_k(0, \chi_{i,n})| \cdot \prod_{i=1}^l |\det(\chi_{i,n}(\delta_i)^{-j})| \\ &= l^{1+v+l^2} (q-1)^{l^2-1} h(\mathcal{O}_F) R_F \prod_{i=1}^l |\det(\chi_{i,n}(\delta_i)^{-j})| \end{aligned}$$

because

$$\prod_{i=0}^l \prod_{n=1}^{l-1} |L_k(0, \chi_{i,n})| = \prod_{\chi_0 \neq \chi \in X_F} |L_k(0, \chi)| = h(F) = h(\mathcal{O}_F) R_F.$$

Let

$$F_i = (\chi_{i,n}(\delta_i)^{-j})_{0 \leq j \leq l-2, 1 \leq n \leq l-1}$$

and

$$H_i = (\chi_i^n(\delta_i)^{-j})_{0 \leq j \leq l-1, 0 \leq n \leq l-1}.$$

Then $\det H_i = l \cdot \det F_i$. Thus

$$|\det F_i| = \frac{1}{l} |\det H_i| = \frac{1}{l} \cdot l^{\frac{l}{2}} = l^{\frac{l-2}{2}}$$

and so

$$|\det(MZ)| = l^{1+v+l^2} (q-1)^{l^2-1} h(\mathcal{O}_F) R_F l^{\frac{(l-2)(l+1)}{2}}$$

Since $|\det Z| = l^{l^2}$ and $|\det M| = l^2 R_B$, we have

$$R_B = l^{v-2+\frac{l(l-1)}{2}} (q-1)^{l^2-1} h(\mathcal{O}_F) R_F.$$

Since $[E : B] = R_B / R_F$, we have

$$[E : B] = l^{v-2+\frac{l(l-1)}{2}} (q-1)^{l^2-1} h(\mathcal{O}_F).$$

□

Finally we compute $[E : C]$ using $[E : B] = [E : C][C : B]$.

THEOREM 3.3. *Let v be the number of subfields of F having an irreducible conductor. Then we have*

$$[E : C] = \begin{cases} l^{v-2+\frac{l(l-1)}{2}} (q-1)^{l^2-1} h(\mathcal{O}_F) & \text{for } k = 0, \\ l^{v-2+\frac{l(l-1)}{2}-a_0} (q-1)^{l^2-1} h(\mathcal{O}_F) & \text{for } k = 1, \\ l^{v-2+\frac{l(l-1)}{2}-a_0-(a_1-1)-(a_2-2)-\dots-(a_b-b)} (q-1)^{l^2-1} h(\mathcal{O}_F) & \text{for } k \geq 2. \end{cases}$$

COROLLARY 3.4. *For the Sinnott index $(R : U)$ we have*

$$(R : U) = \begin{cases} l^{\frac{l(l-1)}{2}} & \text{for } k = 0, \\ l^{\frac{l(l-1)}{2}-a_0} & \text{for } k = 1, \\ l^{\frac{l(l-1)}{2}-a_0-(a_1-1)-(a_2-2)-\dots-(a_b-b)} & \text{for } k \geq 2. \end{cases}$$

Proof. Note that $\prod_{i=1}^s [F_{\mathfrak{p}_i}^{e_i} : k] = l^v$. □

COROLLARY 3.5. *Let r be the number of irreducible factors in the conductor of F and j the number of its nontrivial subfields K_i with*

conductor $f_{K_i} \neq f$, t the number of ramified irreducible polynomials whose decomposition group is bigger than its inertia group. Then,

$$(R : U) = \begin{cases} 1 & \text{if } r = 2, \\ 1 & \text{if } r = 3 \text{ and } j = 2, \\ l & \text{if } r = 3 \text{ and } j = 3, \\ 1 & \text{if } r = 4 \text{ and } j = 2, \\ l & \text{if } r = 4 \text{ and } j = 3, \\ l^2 & \text{if } r = 4 \text{ and } j = 4, t \neq 0, \\ l^3 & \text{if } r = 4 \text{ and } j = 4, t = 0, \end{cases}$$

From the Theorem 3.3, we immediately have the following.

COROLLARY 3.6. *Assume that $l \nmid q - 1$. If $(R : U) = 1, v \leq 1$ or $(R : U) = l, v = 0$, then l divides $h(\mathcal{O}_F)$.*

REMARK 3.1. It is easy to check that the condition of the above corollary is satisfied when $r = 3$ or $(r = 4, j = 2)$, $(r = 4, j = 3)$ in the Corollary 3.5.

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