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PROPERTIES OF NOETHERIAN QUOTIENTS IN R–GROUPS

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Abstract. In this paper, we will introduce the noetherian quotients in R-groups, and then investigate the related substructures of the near-ring R and G and the R-group G.

Also, applying the annihilator concept in R-groups and d.g. near-rings, we will survey some properties of the substructures of R and G in monogenic R-groups and faithful R-groups.

1. Introduction

A near-ring R is an algebraic system $(R, +, \cdot)$ with two binary operations, say $+$ and $\cdot$ such that $(R, +)$ is a group (not necessarily abelian) with neutral element 0, $(R, \cdot)$ is a semigroup and $a(b + c) = ab + ac$ for all a, b, c in R. If R has a unity 1, then R is called *unitary*. An element d in R is called *distributive* if $(a + b)d = ad + bd$ for all a and b in R.

An *ideal* of R is a subset I of R such that (i) $(I,\ +)$ is a normal subgroup of $(R,\ +)$, (ii) $a(I + b) - ab \subset I$ for all $a,\ b \in R$, that is, $RI \subset I$, (iii) $(I + a)b - ab \subset I$ for all $a,\ b \in R$. If I satisfies (i) and (ii) then it is called a *left ideal* of R. If I satisfies (i) and (iii) then it is called a *right ideal* of R.

On the other hand, an *invariant R-subgroup* of R is a subset H of R such that (i) $(H,\ +)$ is a subgroup of $(R,\ +)$, (ii) $RH \subset H$ and (iii) $HR \subset H$. If H satisfies (i) and (ii) then it is called a *left R-subgroup* of R. If H satisfies (i) and (iii) then it is called a *(right) R-subgroup* of R. In case, $(H,\ +)$ is normal in above, we say that *normal invariant R-subgroup*, *normal left R-subgroup* and *normal (right) R-subgroup*, respectively. Note that normal invariant R-subgroups of R are not equivalent to ideals of R.

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We consider the following notations: Given a near-ring R, $R_0 = \{a \in R \mid 0a = 0\}$ which is called the *zero symmetric part* of R, $R_c = \{a \in R \mid 0a = a\} = \{a \in R \mid ra = a,\ for\ all\ r \in R\}$ which is called the *constant part* of R, and $R_d = \{a \in R \mid a\ is\ distributive\}$ which is called the *distributive part* of R.

We note that R_0 and R_c are subnear-rings of R, but R_d is not a subnear-ring of R. A near-ring R with the extra axiom $0a = 0$ for all $a \in R$, that is, $R = R_0$ is said to be *zero symmetric,* also, in case $R = R_c$, R is called a *constant* near-ring, and in case $R = R_d$, R is called a *distributive* near-ring.

From the Pierce decomposition theorem, we get

$$R = R_0 \oplus R_c$$

as additive groups. So every element $a \in R$ has a unique representation of the form $a = b + c$, where $b \in R_0$ and $c \in R_c$.

Let $(G, +)$ be a group (not necessarily abelian). In the set

$$M(G) = \{f \mid f : G \longrightarrow G\}$$

of all the self maps of G, We can define naturally, the sum $f + g$ of any two mappings f, g in $M(G)$ as the rule $x(f + g) = xf + xg$ for all $x \in G$ and the product $f \cdot g$ as the rule $x(f \cdot g) = (xf)g$ for all $x \in G$, then $(M(G), +, \cdot)$ becomes a near-ring. It is called the *self map near-ring* on the group G. Also, we define the set

$$M_0(G) = \{f \in M(G) \mid 0f = 0\},$$

then $(M_0(G), +, \cdot)$ is a zero symmetric near-ring.

Let R and S be two near-rings. Then a mapping θ from R to S is called a *near-ring homomorphism* if (i) $(a + b)\theta = a\theta + b\theta$, (ii) $(ab)\theta = a\theta b\theta$. We can replace homomorphism by momomorphism, epimorphism, isomorphism, endomorphism and automorphism, if these terms have their usual meanings as ring case [1].

Let R be any near-ring and G an additive group. Then G is called an *R-group* if there exists a near-ring homomorphism

$$\theta : (R, +, \cdot) \longrightarrow (M(G), +, \cdot).$$

Such a homomorphism θ is called a *representation* of R on G, we write that xr (right scalar multiplication in R) for $x(r\theta)$ for all $x \in G$ and $r \in R$. If R is unitary, then R-group G is called *unitary*. Thus an R-group is an additive group G satisfying (i) $x(a+b) = xa + xb$, (ii) $x(ab) = (xa)b$ and (iii) $x1 = x$ (If R has a unity 1), for all $x \in G$ and $a,\ b \in R$. Evidently, every near-ring R can be given the structure of an R-group (unitary if R is unitary) by right multiplication in R. Moreover, every group G has a natural $M(G)$-group structure, from the representation of $M(G)$ on G given by applying the $f \in M(G)$ to the $x \in G$ as a scalar multiplication xf.

A representation θ of R on G is called $faithful$ if $Ker\theta = \{0\}$. In this case, we say that G is called a *faithful R-group.*

For an R-group G, a subgroup T of G such that $TR \subset T$ is called an *R-subgroup* of G, a normal subgroup N of G such that $NR \subset N$ is called a *normal R-subgroup* of G and an *R-ideal* of G is a normal subgroup N of G such that $(N + x)a - xa \subset N$ for all $x \in G,\ a \in R$. Also, note that normal R-subgroups of G are not equivalent to an R-ideals of R.

Let R be a near-ring and let G be an R-group. If there exists x in G such that $G = xR$, that is, $G = \{xr \mid r \in R\}$, then G is called a *monogenic R-group* and the element x is called a *generator* of G [7].

For the remainder concepts and notations on near-rings, we refer to Meldrum [6], and Pilz [7].

2. Some results of Noetherian quotients in R-Groups

A near-ring R is called *distributively generated* (briefly, *d.g.*) if it contains a subsemigroup S of $(R_d, \cdot)$ which generates the additive group $(R, +)$, we denote it by (R, S).

On the other hand, the set of all distributive elements of $M(G)$ are obviously the semigroup $End(G)$ of all endomorphisms of the group G under composition. Here we denote that $E(G)$ is the d.g. near-ring generated by $End(G)$, that is, $E(G)$ is d.g. subnear-ring of $(M_0(G), +, \cdot)$ generated by $End(G)$. It is said to be that $E(G)$ is the *endomorphism near-ring* of the group G.

LEMMA 2.1 [5]. *Let (R, S) be a d.g. near-ring. Then all R-subgroups and all R-homomorphic images of a (R, S)-group are also (R, S)-groups.*

Now, we will consider the noetherian quotients which quote frequently, and investigate their related substructures of R and G.

Let G be an R-group and K, K_1 and K_2 be subsets of G. Define

$$(K_1 : K_2) := \{a \in R; K_2 a \subset K_1\}.$$

We abbreviate that for $x \in G$

$$(\{x\} : K_2) =: (x : K_2).$$

Similarly for $(K_1 : x)$. $(0 : K)$ is called the *annihilator* of K, denoted it by $Ann(K)$. We say that G is a *faithful R-group* or that R *acts faithfully* on G if $Ann(G) = \{0\}$, that is, $(0 : G) = \{0\}$.

Also, we see that from the previous concepts to elementwise, a subgroup H of G such that $xa \in H$ for all $x \in H, a \in R$, is an *R-subgroup* of G, and an *R-ideal* of G is a normal subgroup N of G such that

$$(x + g)a - ga \in N$$

for all $g \in G, x \in N$ *and* $a \in R$ (Meldrum [6]).

PROPOSITION 2.2. *Let G be an R-group and K_1 and K_2 subsets of G. Then we have the following conditions:*

(1) *If K_1 is a normal R-subgroup of G, then $(K_1 : K_2)$ is a normal right R- subgroup of R.*

(2) *If* K_1 *is an* R*-subgroup of* G, *then* $(K_1 : K_2)$ *is a right* R*-subgroup of* R.

(3) *If* K_1 *is an* R*-ideal of* G *and* K_2 *is an* R*-subgroup of* G, *then* $(K_1 : K_2)$ *is a two-sided ideal of* R.

Proof. (1) and (2) are proved by Meldrum [6]. Now, we prove only (3) : Using the condition (1), $(K_1 : K_2)$ is a normal subgroup of R. Let $a \in (K_1 : K_2)$ and $r \in R$. Then

$$K_2(ra) = (K_2r)a \subset K_2a \subset K_1,$$

so that $ra \in (K_1 : K_2)$. Whence $(K_1 : K_2)$ is a left ideal of R.

Next, let $r_1, r_2 \in R$ and $a \in (K_1 : K_2)$. Then

$$k\{(a + r_1)r_2 - r_1r_2\} = (ka + kr_1)r_2 - kr_1r_2 \in K_1$$

for all $k \in K_2$, since $K_2a \subset K_1$ and K_1 is an ideal of G. Thus $(K_1 : K_2)$ is a right ideal of R. Therefore $(K_1 : K_2)$ is a two-sided ideal of R. □

COROLLARY 2.3 [6]. *Let* R *be a near-ring and* G *an* R*-group.*

(1) *For any* $x \in G$, $(0 : x)$ *is a right ideal of* R.

(2) *For any* R*-subgroup* K *of* G, $(0 : K)$ *is a two-sided ideal of* R.

(3) *For any subset* K *of* G, $(0 : K) = \bigcap_{x \in K}(0 : x)$.

PROPOSITION 2.4. *Let* R *be a near-ring and* G *an* R*-group. Then we have the following conditions:*

(1) $Ann(G)$ *is an ideal of* R. *Moreover* G *is a faithful* $R/Ann(G)$*-group.*

(2) *For any* $x \in G$, *we get* $xR \cong R/(0 : x)$ *as* R*-groups.*

Proof. (1) By Corollary 2.3 and Proposition 2.2, $Ann(G)$ is a two-sided ideal of R. We now make G an $R/Ann(G)$-group by defining, for $r \in R, r + Ann(G) \in R/Ann(G)$, the action $x(r + Ann(G)) = xr$. If $r + Ann(G) = r' + Ann(G)$, then $-r' + r \in Ann(G)$ hence $x(-r' + r) = 0$ for all x in G, that is to say, $xr = xr'$. This tells us that

$$x(r + Ann(G)) = xr = xr' = x(r' + Ann(G));$$

thus the action of $R/Ann(G)$ on G has been shown to be well defined. The verification of the structure of an $R/Ann(G)$-group is routine. Finally, to see that G is a faithful $R/Ann(G)$-group, we note that if $x(r+Ann(G)) = 0$ for all $x \in G$, then by the definition of $R/Ann(G)$-group structure, we have $xr = 0$. Hence $r \in A(G)$. This says that only the zero element of $R/Ann(G)$ annihilates all of G. Thus G is a faithful $R/Ann(G)$-group.
(2) For any $x \in G$, clearly xR is an R-subgroup of G. The map $\phi : R \longrightarrow xR$ defined by $\phi(r) = xr$ is an R-ephimorphism, so that from the isomorphism theorem, since the kernel of ϕ is $(0 : x)$, we deduce that

$$xR \cong R/(0 : x)$$

as R-groups. □

Corollary 2.5 [7]. *Let G be a monogenic R-group with x as a generator. Then we have the following isomorphic relation.*

$$G \cong R/Ann(x).$$

Proposition 2.6. *If R is a near-ring and G an R-group, then $R/Ann(G)$ is isomorphic to a subnear-ring of $M(G)$.*

Proof. Let $a \in R$. We define $\tau_a : G \longrightarrow G$ by $x\tau_a = xa$ for each $x \in G$. Then τ_a is in $M(G)$. Consider the mapping $\phi : R \longrightarrow M(G)$ defined by $\phi(a) = \tau_a$. Then obviously, we see that

$$\phi(a + b) = \phi(a) + \phi(b) \text{ and } \phi(ab) = \phi(a)\phi(b),$$

that is, ϕ is a near-ring homomorphism from R to $M(G)$.

Next, we must show that $Ker\phi = A(G)$. Indeed, if $a \in Ker\phi$, then $\tau_a = 0$, which implies that $Ga = G\tau_a = 0$, that is, $a \in A(G)$. On the other hand, if $a \in Ann(G)$, then by the definition of $Ann(G)$, $Ga = 0$ hence $0 = \tau_a = \phi(a)$, this implies that $a \in Ker\phi$. Therefore from the first isomorphism theorem on $R-$ groups, the image of R is a near-ring isomorphic to $R/A(G)$. Consequently, $R/A(G)$ is isomorphic to a subnear-ring of $M(G)$. □

Thus we can obtain the following important statement as in ring theory.

COROLLARY 2.7 [7]. *If G is a faithful R-group, then R is embedded in $M(G)$.*

COROLLARY 2.8. *If (R,S) is a d.g. near-ring, then every monogenic R-group is an (R,S)-group.*

Proof. Let G be a monogenic R-group with x as a generator. Then the map $\phi : r \mapsto\ xr$ is an R-epimorphism from R to G as R-groups. We see that by the Corollary 2.5,

$$G \cong R/A(x),$$

where $Ann(x) = Ker\phi$. From the Lemma 2.1, we see that G is an (R,S)-group. □

Now, we get the following useful results of monogenic R-groups.

PROPOSITION 2.9. *Let G be a monogenic R-group with generator x. Then we have the following properties:*

(1) *For any right ideal I of R, xI is an R-ideal of G.*
(2) *If I is a left R-subgroup of R and xI is an R-ideal of G, then I is an ideal of R.*
(3) *If e is a right identity of R and if G is a faithful R-group, then e is a two-sided identity of R.*

Proof. (1) Let $a \in G$. Then there exists $t \in R$ such that $a = xt$. Thus for each $xy \in xI, r \in R$, and $a \in G$,

$$(a + xy)r - ar = (xt + xy)r - (xt)r = x(t + y)r - x(tr)$$

$$= x\{(t + y)r - tr\} \in xI$$

By using similar method, it can be easily shown that xI is an additive normal subgroup of G. Therefore xI is an R-ideal of G.
(2) For any $y \in I$ and $a,\ b \in R$, we obtain the following equality:

$$x\{(y+a)b-ab\} = x(y+a)b - x(ab) = (xy+xa)b - (xa)b) \in xI$$

Hence $(y+a)b - ab \in xI$. Similarly, we can show that I is an additive normal subgroup of R. Consequently, I is an ideal of R.

(3) First, let e be a right identity of R and $g = xt$ be any element in G. Then we have the relation that

$$ge = (xt)e = x(te) = xt = g$$

Next, let r be any element of R and g be an arbitrary element in G. Then one gets the following equality that

$$g(er - r) = g(er) + g(-r) = (ge)r - gr = gr - gr = 0$$

Thus $(er - r) \in Ann(G)$.

Since G is faithful, it implies that $er - r = 0$, that is, $er = r$. Hence e is a two-sided identity of R. □

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