

GENERALIZED CUBIC MAPPINGS OF r - TYPE IN SEVERAL VARIABLES

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ABSTRACT. Let X, Y be vector spaces. In this paper, we investigate the generalized Hyers-Ulam-Rassias stability problem for a cubic function $f : X \rightarrow Y$ satisfies

$$\begin{aligned} & r^3 f\left(\frac{\sum_{j=1}^{n-1} x_j + 2x_n}{r}\right) + r^3 f\left(\frac{\sum_{j=1}^{n-1} x_j - 2x_n}{r}\right) + 8 \sum_{j=1}^{n-1} f(x_j) \\ &= 2f\left(\sum_{j=1}^{n-1} x_j\right) + 4 \sum_{j=1}^{n-1} \left(f(x_j + x_n) + f(x_j - x_n)\right) \end{aligned}$$

for all $x_1, \dots, x_n \in X$.

1. Introduction

The study of stability problems for functional equations is related to the following question originated by Ulam [13] concerning the stability of group homomorphisms:

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$ does there exist a $\delta > 0$ such that if a mapping $h : G_1 \rightarrow G_2$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta$$

for all $x, y \in G_1$, then a homomorphism $H : G_1 \rightarrow G_2$ exists with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The first partial solution to Ulam's question was provided by D. H. Hyers [5]. Let X and Y are Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Hyers showed that if a function $f : X \rightarrow Y$ satisfies the following inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

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for all $\epsilon \geq 0$ and for all $x, y \in X$, then the limit

$$a(x) = \lim_{n \rightarrow \infty} 2^{-n} f(2^n x)$$

exists for each $x \in X$ and $a : X \rightarrow Y$ is the unique additive function such that

$$\|f(x) - a(x)\| \leq \epsilon$$

for any $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then a is linear.

Hyers's theorem was generalized in various directions. In particular, Th. M. Rassias [8] and Z. Gajda [3] considered a generalized version of the theorem of Hyers which permitted the Cauchy difference to become unbounded. They proved the following theorem by using a direct method: if a function $f : X \rightarrow Y$ satisfies the following inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for some $\theta \geq 0$, $0 \leq p < 1$, and for all $x, y \in X$, then there exists a unique additive function such that

$$\|f(x) - a(x)\| \leq \frac{2\theta}{2-2^p} \|x\|^p$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then a is linear. Găvruta [4] generalized the Rassias's result above.

The quadratic function $f(x) = cx^2$ ($c \in \mathbb{R}$) satisfies the functional equation

$$(1.1) \quad f(x+y) + f(x-y) = 2f(x) + 2f(y).$$

This question is called the quadratic functional equation, and every solution of the equation (1.1) is called a quadratic function.

A Hyers-Ulam stability theorem for the quadratic functional equation (1.1) was first proved by Skof [12] for functions $f : X \rightarrow Y$, where X is a normed space and Y is a Banach space. Cholewa [1] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an abelian group. In [2], Czerwik proved the Hyers-Ulam-Rassias stability of the quadratic functional equation. Several functional equations have been investigated; see [9], [10], and [11].

The cubic function $f(x) = cx^3$ ($c \in \mathbb{R}$) satisfies the functional equation

$$(1.2) \quad f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x).$$

We promise that by a cubic function we mean every solution of the equation (1.2) is called a cubic function. The equation (1.2) was solved

by Jun and Kim [6]. Also, they proved the generalized Hyers-Ulam-Rassias stability problem for the given functional equation : see [7]

$$(1.3) \quad f(x+2y) + f(x-2y) + 6f(x) = 4f(x+y) + 4f(x-y).$$

Throughout this paper, we assume that $r(r^3 \neq 1)$ is a real number and $n \geq 2$ is an integer number.

In this paper, for all $x_1, \dots, x_n \in X$ the following odd functional equation $f : X \rightarrow Y$ such that

$$(1.4) \quad r^3 f\left(\frac{\sum_{j=1}^{n-1} x_j + 2x_n}{r}\right) + r^3 f\left(\frac{\sum_{j=1}^{n-1} x_j - 2x_n}{r}\right) + 8 \sum_{j=1}^{n-1} f(x_j) \\ = 2f\left(\sum_{j=1}^{n-1} x_j\right) + 4 \sum_{j=1}^{n-1} \left(f(x_j + x_n) + f(x_j - x_n)\right)$$

is a solution of the equation (1.3) and then investigate the generalized Hyers-Ulam-Rassias stability in Banach spaces.

2. Generalized cubic mappings of r -type in several variables

LEMMA 2.1. *Let X and Y be real vector spaces. If an odd function $f : X \rightarrow Y$ satisfies the functional equation (1.4), then there exists functions $B : X \times X \times X \rightarrow Y$ and $A : X \rightarrow Y$ such that*

$$B(x, y, z) = \frac{1}{24} [f(x+y+z) + f(x-y-z) - f(x+y-z) - f(x-y+z)]$$

is symmetric for each fixed one variable and additive for each two variables and A is additive.

Proof. Since f is odd, we have $f(0) = 0$. Now, setting $x_1 = x, x_n = y$, and $x_j = 0$ ($j = 2, \dots, n-1$), we write

$$(2.1) \quad r^3 f\left(\frac{x+2y}{r}\right) + r^3 f\left(\frac{x-2y}{r}\right) + 6f(x) = 4f(x+y) + 4f(x-y),$$

for all $x, y \in X$. By letting $y = 0$, we have

$$r^3 f\left(\frac{x}{r}\right) = f(x),$$

for all $x \in X$. Thus we may conclude that

$$\begin{aligned} 4f(x+y) + 4f(x-y) &= r^3 f\left(\frac{x+2y}{r}\right) + r^3 f\left(\frac{x-2y}{r}\right) + 6f(x) \\ &= f(x+2y) + f(x-2y) + 6f(x), \end{aligned}$$

for all $x, y \in X$, that is, the equation (1.4) satisfies the equation (1.3). The remains of proof follow from [7, Theorem 2.1]. $\square$

LEMMA 2.2. *Let X and Y be real vector spaces. If an odd function $f : X \rightarrow Y$ satisfies the functional equation (1.3) and $f(2x) = 8f(x)$, for all $x \in X$, then f is cubic.*

Proof. Letting $x = 2x$ in the equation (1.3), we have

$$f(2x + 2y) + f(2x - 2y) + 6f(2x) = 4f(2x + y) + 4f(2x - y).$$

Since $f(2x) = 8f(x)$, for all $x \in X$, we get

$$8f(x + y) + 8f(x - y) + 48f(x) = 4f(2x + y) + 4f(2x - y),$$

that is, it satisfies the equation (1.2), as desired. $\square$

3. Hyers-Ulam-Rassias stability

Throughout in this section, let X be a normed vector space with norm $\|\cdot\|$ and Y be a Banach space with norm $\|\cdot\|$. For the given odd mapping $f : X \rightarrow Y$, we define

$$(3.1) \quad Df(x_1, \dots, x_n) := r^3 f\left(\frac{\sum_{j=1}^{n-1} x_j + 2x_n}{r}\right) + r^3 f\left(\frac{\sum_{j=1}^{n-1} x_j - 2x_n}{r}\right) \\ + 8 \sum_{j=1}^{n-1} f(x_j) - 2f\left(\sum_{j=1}^{n-1} x_j\right) - 4 \sum_{j=1}^{n-1} \left(f(x_j + x_n) + f(x_j - x_n)\right),$$

for all $x_1, \dots, x_n \in X$.

THEOREM 3.1. *Let $n \geq 2$ be an integer number, let $|r| < 1$, and let $f : X \rightarrow Y$ be an odd mapping for which there exists a function $\phi : X^n \rightarrow [0, \infty)$ such that*

$$(3.2) \quad \tilde{\phi}(x_1, \dots, x_n) := \sum_{j=0}^{\infty} r^{3j} \phi\left(\frac{x_1}{r^j}, \dots, \frac{x_n}{r^j}\right) < \infty,$$

$$(3.3) \quad \|Df(x_1, \dots, x_n)\| \leq \phi(x_1, \dots, x_n),$$

and

$$\|f(2x) - 8f(x)\| \leq \delta$$

for all $x_1, \dots, x_n, x \in X$ and for some $\delta \geq 0$. Then for every $m \in \{1, 2, \dots, n-1\}$, there exists a generalized cubic mapping $C : X \rightarrow Y$ such that

$$(3.4) \quad \|f(x) - C(x)\| \leq \underbrace{\tilde{\phi}\left(\frac{x}{m}, \dots, \frac{x}{m}, 0, \dots, 0\right)}_{m\text{-terms}},$$

for all $x \in X$.

Proof. Let $1 \leq m \leq n-1$ be an integer number. By setting $x_j = \frac{x}{m}$ ($j = 1, \dots, m$) and $x_{m+1} = \dots = x_n = 0$, we have

$$(3.5) \quad \|f(x) - r^3 f\left(\frac{x}{r}\right)\| \leq \underbrace{\phi\left(\frac{x}{m}, \dots, \frac{x}{m}, 0, \dots, 0\right)}_m,$$

for all $x \in X$. Replacing x by $\frac{x}{m}$ in the equation (3.5) and then multiplying by r^3 , we have

$$(3.6) \quad \|r^3 f\left(\frac{x}{r}\right) - r^{3 \cdot 2} f\left(\frac{x}{r^2}\right)\| \leq r^3 \underbrace{\phi\left(\frac{x}{m}, \dots, \frac{x}{m}, 0, \dots, 0\right)}_m,$$

for all $x \in X$. Now, combining equations (3.5) and (3.6), we get

$$\begin{aligned} \|f(x) - r^{3 \cdot 2} f\left(\frac{x}{r^2}\right)\| &\leq \underbrace{\phi\left(\frac{x}{m}, \dots, \frac{x}{m}, 0, \dots, 0\right)}_m \\ &+ r^3 \underbrace{\phi\left(\frac{x}{m \cdot r}, \dots, \frac{x}{m \cdot r}, 0, \dots, 0\right)}_m, \end{aligned}$$

for all $x \in X$. Continue this way, we may have

$$(3.7) \quad \|f(x) - r^{3 \cdot j} f\left(\frac{x}{r^j}\right)\| \leq \sum_{k=0}^{j-1} r^{3 \cdot k} \underbrace{\phi\left(\frac{x}{m \cdot r^k}, \dots, \frac{x}{m \cdot r^k}, 0, \dots, 0\right)}_m,$$

for all positive integer j and all $x \in X$.

Dividing the equation (3.7) by $r^{3 \cdot s}$ and then substituting x by $\frac{x}{r^s}$, we have

$$\begin{aligned} &r^{3 \cdot s} \|f\left(\frac{x}{r^s}\right) - r^{3 \cdot j} f\left(\frac{x}{r^{j+s}}\right)\| \\ &\leq r^{3 \cdot s} \sum_{k=0}^{j-1} r^{3 \cdot k} \underbrace{\phi\left(\frac{x}{m \cdot r^{k+s}}, \dots, \frac{x}{m \cdot r^{k+s}}, 0, \dots, 0\right)}_m, \end{aligned}$$

for all $x \in X$.

By taking $s \rightarrow \infty$, we may conclude that $\{r^{3 \cdot j} f(\frac{x}{r^j})\}$ is a Cauchy sequence in a Banach space Y . This implies that the sequence $\{r^{3 \cdot j} f(\frac{x}{r^j})\}$ converges. Hence we can define a function $C : X \rightarrow Y$ by

$$C(x) = \lim_{j \rightarrow \infty} r^{3 \cdot j} f\left(\frac{x}{r^j}\right),$$

for all $x \in X$. Then

$$\begin{aligned} \|DC(x_1, \dots, x_n)\| &= \lim_{k \rightarrow \infty} r^{3 \cdot k} \|Df\left(\frac{x_1}{r^k}, \dots, \frac{x_n}{r^k}\right)\| \\ &\leq \lim_{k \rightarrow \infty} r^{3 \cdot k} \phi\left(\frac{x_1}{r^k}, \dots, \frac{x_n}{r^k}\right) \\ &= 0, \end{aligned}$$

for all $x_1, \dots, x_n \in X$. That is, $DC(x_1, \dots, x_n) = 0$. Obviously, we have C is odd and $C(2x) = 8C(x)$, for all $x \in X$. Corollary 2.2 implies that the function $C : X \rightarrow Y$ is a cubic mapping.

It only remains to show that the function C is unique. Let $C' : X \rightarrow Y$ be another generalized cubic function satisfying the equation (3.4). Then

$$\begin{aligned} \|C(x) - C'(x)\| &= r^{3k} \|C((\frac{1}{r})^k x) - C'((\frac{1}{r})^k x)\| \\ &\leq r^{3k} \left(\|C((\frac{1}{r})^k x) - f((\frac{1}{r})^k x)\| + \|C'((\frac{1}{r})^k x) - f((\frac{1}{r})^k x)\| \right) \\ &\leq 2 \cdot r^{3k} \tilde{\phi}\left((\frac{1}{r})^k x, (\frac{1}{r})^k x, 0, \dots, 0\right) \\ &= 2 \cdot r^{3k} \sum_{j=0}^{\infty} r^{3j} \phi\left((\frac{1}{r})^{j+k} x, (\frac{1}{r})^{j+k} x, 0, \dots, 0\right) \\ &= 2 \cdot \sum_{j=k}^{\infty} r^{3j} \phi\left((\frac{1}{r})^j x, (\frac{1}{r})^j x, 0, \dots, 0\right) \rightarrow 0, \end{aligned}$$

for all $x \in X$. As $r \rightarrow \infty$, we can conclude that $C(x) = C'(x)$, for all $x \in X$; that is, C is unique. $\square$

COROLLARY 3.2. *Let $r > 1$, and let θ and $p < 3$ be positive real numbers (Or, let $r < 1$, and let θ and $p > 3$ be positive real numbers).*

Let $f : X \rightarrow Y$ be a function such that

$$\|Df(x_1, \dots, x_n)\| \leq \theta \sum_{i=1}^n \|x_i\|^p,$$

for all $x_1, \dots, x_n \in X$. Then there exists a unique generalized cubic mapping $C : X \rightarrow Y$ such that

$$\|f(x) - C(x)\| \leq \frac{m\theta}{r^{3-p} - 1} \|x\|^p,$$

for all $x \in X$, and for any $1 \leq m \leq n - 1$.

Proof. Define $\phi(x_1, \dots, x_n) = \theta \sum_{i=1}^n \|x_i\|^p$, and apply to Theorem 3.1. $\square$

THEOREM 3.3. Let $n \geq 2$ be an integer number, let $|r| > 1$, and let $f : X \rightarrow Y$ be a mapping for which there exists a function $\phi : X^n \rightarrow [0, \infty)$ such that

$$(3.8) \quad \tilde{\phi}(x_1, \dots, x_n) := \sum_{j=1}^{\infty} r^{-3j} \phi(r^j x_1, \dots, r^j x_n) < \infty,$$

$$(3.9) \quad \|Df(x_1, \dots, x_n)\| \leq \phi(x_1, \dots, x_n),$$

and

$$\|f(2x) - 8f(x)\| \leq \delta$$

for all $x_1, \dots, x_n, x \in X$ and for some $\delta \geq 0$. Then for every $m \in \{1, 2, \dots, n - 1\}$, there exists a generalized cubic mapping $C : X \rightarrow Y$ such that

$$(3.10) \quad \|f(x) - C(x)\| \leq \tilde{\phi}\left(\underbrace{\frac{x}{m}, \dots, \frac{x}{m}}_{m\text{-terms}}, 0, \dots, 0\right),$$

for all $x \in X$.

Proof. If x is replaced by rx and dividing by r^3 in the equation (3.5) in the proof of Theorem 3.1, we have the following equation

$$(3.11) \quad \left\|f(x) - \frac{1}{r^3} f(rx)\right\| \leq \frac{1}{r^3} \phi\left(\underbrace{\frac{r}{m}x, \dots, \frac{r}{m}x}_m, 0, \dots, 0\right),$$

for all $x \in X$. The remains of the proof are similar to the proof of Theorem 3.1. $\square$

COROLLARY 3.4. Let $r > 1$, and let θ and $p > 3$ be positive real numbers (Or, let $r < 1$, and let θ and $p < 3$ be positive real numbers). Let $f : X \rightarrow Y$ be a function such that

$$\|Df(x_1, \dots, x_n)\| \leq \theta \sum_{i=1}^n \|x_i\|^p,$$

for all $x_1, \dots, x_n \in X$. Then there exists a unique generalized cubic mapping $C : X \rightarrow Y$ such that

$$\|f(x) - C(x)\| \leq m\theta \cdot \frac{r^{p-3}}{r^{3-p} - 1} \|x\|^p,$$

for all $x \in X$, and for any $1 \leq m \leq n - 1$.

Proof. Define $\phi(x_1, \dots, x_n) = \theta \sum_{i=1}^n \|x_i\|^p$, and apply to Theorem 3.3. $\square$

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