

ON THE HYERS-ULAM-RASSIAS STABILITY OF A CAUCHY-JENSEN FUNCTIONAL EQUATION

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ABSTRACT. In this paper, we prove the Hyers-Ulam-Rassias stability of a Cauchy-Jensen functional equation

$$2f(x+y, \frac{z+w}{2}) = f(x, z) + f(x, w) + f(y, z) + f(y, w).$$

1. Introduction

In 1940, S. M. Ulam [5] raised a question concerning the stability of homomorphisms: Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \rightarrow G_2$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta$$

for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \rightarrow G_2$ with

$$d(h(x), H(x)) < \varepsilon$$

for all $x \in G_1$? The case of approximately additive mappings was solved by Hyers [2] under the assumption that G_1 and G_2 are Banach spaces. In 1978, Rassias [4] gave a generalization. Recently, Găvruta [1] also obtained a further generalization of the Hyers-Ulam-Rassias theorem.

Throughout this paper, let X be normed space and Y be a Banach space. A mapping $g : X \rightarrow Y$ is called a Cauchy mapping (respectively, a Jensen mapping) if g satisfies the functional equation $g(x+y) = g(x) + g(y)$ (respectively, $2g(\frac{x+y}{2}) = g(x) + g(y)$).

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A mapping $f : X \times X \rightarrow Y$ is called a Cauchy-Jensen mapping[3] if f satisfies the system of equations

$$(1.1) \quad \begin{aligned} f(x+y, z) &= f(x, z) + f(y, z), \\ 2f(x, \frac{y+z}{2}) &= f(x, y) + f(x, z). \end{aligned}$$

It is easy to see that a mapping $f : X \times X \rightarrow Y$ is a Cauchy-Jensen mapping if and only if the mapping f satisfies the functional equation

$$(1.2) \quad 2f(x+y, \frac{z+w}{2}) = f(x, z) + f(x, w) + f(y, z) + f(y, w)$$

for all $x, y, z, w \in X$. In 2006, Park and Bae [3] obtained the generalized Hyers-Ulam stability of (1) and (2). In this paper, I study the Hyers-Ulam-Rassias stability of (1) and (2).

2. Stability of (1) and (2)

For the given mapping $f : X \times X \rightarrow Y$, we define

$$\begin{aligned} Df(x, y, z, w) &:= 2f(x+y, \frac{z+w}{2}) - f(x, z) \\ &\quad - f(x, w) - f(y, z) - f(y, w) \\ D_1f(x, y, z) &:= f(x+y, z) - f(x, z) - f(y, z), \\ D_2f(x, y, z) &:= 2f(x, \frac{y+z}{2}) - f(x, y) - f(x, z). \end{aligned}$$

for all $x, y, z, w \in X$.

THEOREM 2.1. *Let $p_1, p_2, p_3, q_1, q_2, q_3, \delta_1, \delta_2, \delta_3, \delta_4$ be fixed positive real numbers with $p_1, p_2, p_3, q_1, q_2, q_3 < 1$. Let $f : X \times X \rightarrow Y$ be a mapping such that*

$$(2.1) \quad \|D_1f(x, y, z)\| \leq (\|x\|^{p_1} + \|y\|^{p_2} + \delta_1)(\|z\|^{q_1} + \delta_2),$$

$$(2.2) \quad \|D_2f(x, y, z)\| \leq (\|x\|^{p_3} + \delta_3)(\|y\|^{q_2} + \|z\|^{q_3} + \delta_4)$$

for all $x, y, z \in X$. Then there exist a unique Cauchy-Jensen mapping $F_1 : X \times X \rightarrow Y$ and a unique biadditive mapping $F_2 : X \times X \rightarrow Y$ such that

$$(2.3) \quad \|f(x, y) - F_1(x, y)\| \leq (\frac{\|x\|^{p_1}}{2-2^{p_1}} + \frac{\|x\|^{p_2}}{2-2^{p_2}} + \delta_1)(\|y\|^{q_1} + \delta_2),$$

$$(2.4) \quad \|f(x, y) - f(x, 0) - F_2(x, y)\| \leq (\|x\|^{p_3} + \delta_3)(\frac{2^{q_2}}{2-2^{q_2}}\|y\|^{q_2} + \delta_4),$$

$$(2.5) \quad F_1(x, y) - F_1(x, 0) = F_2(x, y)$$

for all $x, y \in X$. The mappings $F_1, F_2 : X \times X \rightarrow Y$ are given by

$$F_1(x, y) := \lim_{j \rightarrow \infty} \frac{1}{2^j} f(2^j x, y), \quad F_2(x, y) := \lim_{j \rightarrow \infty} \frac{1}{2^j} f(x, 2^j y)$$

for all $x, y \in X$.

Proof. Letting $y = x$ and replacing z by y in (2.1),

$$\|f(2x, y) - 2f(x, y)\| \leq (\|x\|^{p_1} + \|x\|^{p_2} + \delta_1)(\|y\|^{q_1} + \delta_2)$$

for all $x, y \in X$. Thus

$$\begin{aligned} & \left\| \frac{1}{2^j} f(2^j x, y) - \frac{1}{2^{j+1}} f(2^{j+1} x, y) \right\| \\ & \leq \left(\frac{2^{jp_1}}{2^{j+1}} \|x\|^{p_1} + \frac{2^{jp_2}}{2^{j+1}} \|x\|^{p_2} + \frac{\delta_1}{2^{j+1}} \right) (\|y\|^{q_1} + \delta_2) \end{aligned}$$

for all $x, y \in X$. For given integers l, m ($0 \leq l < m$),

$$\begin{aligned} & \left\| \frac{1}{2^l} f(2^l x, y) - \frac{1}{2^m} f(2^m x, y) \right\| \\ (2.6) \quad & \leq \sum_{j=l}^{m-1} \left(\frac{2^{jp_1}}{2^{j+1}} \|x\|^{p_1} + \frac{2^{jp_2}}{2^{j+1}} \|x\|^{p_2} + \frac{\delta_1}{2^{j+1}} \right) (\|y\|^{q_1} + \delta_2) \end{aligned}$$

for all $x, y \in X$. By $p_1, p_2 < 1$, the sequence $\{\frac{1}{2^j} f(2^j x, y)\}$ is a Cauchy sequence for all $x, y \in X$. Since Y is complete, the sequence $\{\frac{1}{2^j} f(2^j x, y)\}$ converges for all $x, y \in X$. Define $F_1 : X \times X \rightarrow Y$ by

$$F_1(x, y) := \lim_{j \rightarrow \infty} \frac{1}{2^j} f(2^j x, y)$$

for all $x, y \in X$. Putting $l = 0$ and taking $m \rightarrow \infty$ in (2.6), one can obtain the inequality (2.3). By (2.1) and (2.2),

$$\begin{aligned} \left\| \frac{1}{2^j} D_1 f(2^j x, 2^j y, z) \right\| & \leq \left(\frac{2^{jp_1}}{2^j} \|x\|^{p_1} + \frac{2^{jp_2}}{2^j} \|y\|^{p_2} + \frac{\delta_1}{2^j} \right) (\|z\|^{q_1} + \delta_2), \\ \left\| \frac{1}{2^j} D_2 f(2^j x, y, z) \right\| & \leq \left(\frac{2^{jp_3}}{2^j} \|x\|^{p_3} + \frac{\delta_3}{2^j} \right) (\|y\|^{q_2} + \|z\|^{q_3} + \delta_4) \end{aligned}$$

for all $x, y, z \in X$ and all j . Letting $j \rightarrow \infty$ in the above two inequalities and using $p_1, p_2, p_3 < 1$, F_1 is a Cauchy-Jensen mapping. Now, let $F'_1 : X \times X \rightarrow Y$ be another Cauchy-Jensen mapping satisfying (2.3).

Then we have

$$\begin{aligned}
& \|F_1(x, y) - F'_1(x, y)\| \\
& \leq \frac{1}{2^n} \|f(2^n x, y) - F_1(2^n x, y)\| + \frac{1}{2^n} \|f(2^n x, y) - F'_1(2^n x, y)\| \\
& \leq \left(\frac{2^{np_1}}{2^n} \frac{\|x\|^{p_1}}{2 - 2^{p_1}} + \frac{2^{np_2}}{2^n} \frac{\|x\|^{p_2}}{2 - 2^{p_2}} + \frac{\delta_1}{2^n} \right) (\|y\|^{q_1} + \delta_2)
\end{aligned}$$

for all $n \in N$ and $x, y \in X$. As $n \rightarrow \infty$, we may conclude that $F_1(x, y) = F'_1(x, y)$ for all $x, y \in X$. Thus such a Cauchy-Jensen mapping $F_1 : X \times X \rightarrow Y$ is unique.

Next, replacing y by $2y$ and z by 0 in (2.2), one can obtain

$$\begin{aligned}
(2.7) \quad & \|(f(x, y) - f(x, 0)) - \frac{1}{2}(f(x, 2y) - f(x, 0))\| \\
& \leq \frac{1}{2} (\|x\|^{p_3} + \delta_3) (2^{q_2} \|y\|^{q_2} + \|0\|^{q_3} + \delta_4)
\end{aligned}$$

for all $x, y \in X$. By the same method as above, F_2 is a unique biadditive mapping which satisfies (2.4), where $F_2(x, y) := \lim_{j \rightarrow \infty} \frac{1}{2^j} f(x, 2^j y)$ for all $x, y \in X$. From (2.7) and the definitions of F_1 and F_2 , the equalities

$$\begin{aligned}
(2.8) \quad & F_1(x, y) - F_1(x, 0) = \frac{1}{2} (F_1(x, 2y) - F_1(x, 0)), \\
& F_1(x, y) - F_1(x, 0) = \frac{1}{2^n} F_1(x, 2^n y) - \frac{1}{2^n} F_1(x, 0), \\
& F_2(x, y) = \frac{1}{2^n} F_2(x, 2^n y)
\end{aligned}$$

hold for all $n \in N$ and for all $x, y \in X$. Hence, the inequality

$$\begin{aligned}
& \|F_1(x, y) - F_1(x, 0) - F_2(x, y)\| \\
& = \left\| \frac{1}{2^n} F_1(x, 2^n y) - \frac{1}{2^n} F_1(x, 0) - \frac{1}{2^n} F_2(x, 2^n y) \right\| \\
& = \frac{1}{2^n} \|f(x, 2^n y) - F_1(x, 2^n y)\| + \frac{1}{2^n} \|f(x, 0) - F_1(x, 0)\| \\
& \quad + \frac{1}{2^n} \|f(x, 2^n y) - f(x, 0) - F_2(x, 2^n y)\| \\
& \leq \left(\frac{\|x\|^{p_1}}{2 - 2^{p_1}} + \frac{\|x\|^{p_2}}{2 - 2^{p_2}} + \delta_1 \right) \left(\frac{2^{nq_1}}{2^n} \|y\|^{q_1} + \frac{2\delta_2}{2^n} \right) \\
& \quad + (\|x\|^{p_3} + \delta_3) \left(\frac{2^{nq_2}}{2^n} \frac{2^{q_2}}{2 - 2^{q_2}} \|y\|^{q_2} + \frac{1}{2^n} \delta_4 \right)
\end{aligned}$$

holds for all $n \in N$ and for all $x, y \in X$. Taking $n \rightarrow \infty$ and using $q_1, q_2 < 1$, we have (2.5). $\square$

THEOREM 2.2. *Let $p_1, p_2, p_3, q_1, q_2, q_3$ be fixed positive real numbers with $1 < p_1, p_2, p_3, q_1, q_2, q_3$. Let $f : X \times X \rightarrow Y$ be a mapping such that*

$$(2.9) \quad \|D_1 f(x, y, z)\| \leq (\|x\|^{p_1} + \|y\|^{p_2})\|z\|^{q_1},$$

$$(2.10) \quad \|D_2 f(x, y, z)\| \leq \|x\|^{p_3}(\|y\|^{q_2} + \|z\|^{q_3})$$

for all $x, y, z \in X$. Then there exist a unique Cauchy-Jensen mapping $F_1 : X \times X \rightarrow Y$ and a unique biadditive mapping $F_2 : X \times X \rightarrow Y$ such that

$$(2.11) \quad \|f(x, y) - F_1(x, y)\| \leq \left(\frac{\|x\|^{p_1}}{2^{p_1} - 2} + \frac{\|x\|^{p_2}}{2^{p_2} - 2}\right)\|y\|^{q_1},$$

$$(2.12) \quad \|f(x, y) - f(x, 0) - F_2(x, y)\| \leq \|x\|^{p_3} \frac{2^{q_2}}{2^{q_2} - 2} \|y\|^{q_2},$$

$$(2.13) \quad F_1(x, y) - f(x, 0) = F_2(x, y)$$

for all $x, y \in X$. The mappings $F_1, F_2 : X \times X \rightarrow Y$ are given by

$$F_1(x, y) := \lim_{j \rightarrow \infty} 2^j f\left(\frac{x}{2^j}, y\right), \quad F_2(x, y) := \lim_{j \rightarrow \infty} 2^j (f(x, \frac{y}{2^j}) - f(x, 0))$$

for all $x, y \in X$.

Proof. Replacing x, y, z by $\frac{x}{2}, \frac{x}{2}, y$ in (2.9) respectively, we have

$$\|f(x, y) - 2f\left(\frac{x}{2}, y\right)\| \leq (\|\frac{x}{2}\|^{p_1} + \|\frac{x}{2}\|^{p_2})\|y\|^{q_1}$$

for all $x, y \in X$. Thus

$$\|2^j f\left(\frac{x}{2^j}, y\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}, y\right)\| \leq \left(\left(\frac{2}{2^{p_1}}\right)^j \|\frac{x}{2}\|^{p_1} + \left(\frac{2}{2^{p_2}}\right)^j \|\frac{x}{2}\|^{p_2}\right)\|y\|^{q_1}$$

for all $x, y \in X$. For given integers l, m ($0 \leq l < m$),

$$(2.14) \quad \begin{aligned} & \|2^l f\left(\frac{x}{2^l}, y\right) - 2^m f\left(\frac{x}{2^m}, y\right)\| \\ & \leq \sum_{j=l}^{m-1} \left(\left(\frac{2}{2^{p_1}}\right)^j \|\frac{x}{2}\|^{p_1} + \left(\frac{2}{2^{p_2}}\right)^j \|\frac{x}{2}\|^{p_2}\right)\|y\|^{q_1} \end{aligned}$$

for all $x, y \in X$. By (2.14), the sequence $\{2^j f(\frac{x}{2^j}, y)\}$ is a Cauchy sequence for all $x, y \in X$. Since Y is complete, the sequence $\{2^j f(\frac{x}{2^j}, y)\}$ converges for all $x, y \in X$. Define $F_1 : X \times X \rightarrow Y$ by

$$F_1(x, y) := \lim_{j \rightarrow \infty} 2^j f\left(\frac{x}{2^j}, y\right)$$

for all $x, y \in X$. Putting $l = 0$ and taking $m \rightarrow \infty$ in (2.14), one can obtain the inequality (2.11). By (2.9) and (2.10),

$$\begin{aligned} \|2^j D_1 f(\frac{x}{2^j}, \frac{y}{2^j}, z)\| &\leq \left((\frac{2}{2^{p_1}})^j \|x\|^{p_1} + (\frac{2}{2^{p_2}})^j \|y\|^{p_2} \right) \|z\|^{q_1}, \\ \|2^j D_2 f(\frac{x}{2^j}, y, z)\| &\leq (\frac{2}{2^{p_3}})^j \|x\|^{p_3} (\|y\|^{q_2} + \|z\|^{q_3}) \end{aligned}$$

for all $x, y, z \in X$ and all j . Letting $j \rightarrow \infty$ in the above two inequalities and using $1 < p_1, p_2, p_3$, F_1 is a Cauchy-Jensen mapping. Now, let $F'_1 : X \times X \rightarrow Y$ be another Cauchy-Jensen mapping satisfying (2.11). Then we have

$$\begin{aligned} \|F_1(x, y) - F'_1(x, y)\| &\leq 2^n \|f(\frac{x}{2^n}, y) - F_1(\frac{x}{2^n}, y)\| + 2^n \|f(\frac{x}{2^n}, y) - F'_1(\frac{x}{2^n}, y)\| \\ &\leq \left((\frac{2}{2^{p_1}})^n \frac{2\|x\|^{p_1}}{2^{p_1} - 2} + (\frac{2}{2^{p_2}})^n \frac{2\|x\|^{p_2}}{2^{p_2} - 2} \right) \|y\|^{q_1} \end{aligned}$$

for all $n \in N$ and $x, y \in X$. As $n \rightarrow \infty$, we may conclude that $F_1(x, y) = F'_1(x, y)$ for all $x, y \in X$. Thus such a Cauchy-Jensen mapping $F_1 : X \times X \rightarrow Y$ is unique.

Next, replacing z by 0 in (2.10), one can obtain

$$(2.15) \quad \|(f(x, y) - f(x, 0)) - 2(f(x, \frac{y}{2}) - f(x, 0))\| \leq \|x\|^{p_3} \|y\|^{q_2}$$

for all $x, y \in X$. By the same method as above, F_2 is a unique biadditive mapping which satisfies (2.12), where $F_2(x, y) := \lim_{j \rightarrow \infty} 2^j (f(x, \frac{y}{2^j}) - f(x, 0))$ for all $x, y \in X$. From (2.15) and the definitions of F_1 and F_2 , the equalities

$$\begin{aligned} F_1(x, y) - F_1(x, 0) &= 2(F_1(x, \frac{y}{2}) - F_1(x, 0)), \\ F_1(x, y) - F_1(x, 0) &= 2^n F_1(x, \frac{y}{2^n}) - 2^n F_1(x, 0), \\ (2.16) \quad F_2(x, y) &= 2^n F_2(x, \frac{y}{2^n}) \end{aligned}$$

hold for all $n \in N$ and for all $x, y \in X$. By (2.11), the equality $f(x, 0) = F_1(x, 0)$ holds for all $x \in X$. Hence, by (2.11), (2.12) and the above

equalities, the inequality

$$\begin{aligned}
 & \|F_1(x, y) - F_1(x, 0) - F_2(x, y)\| \\
 &= \|2^n F_1(x, \frac{y}{2^n}) - 2^n F_1(x, 0) - 2^n F_2(x, \frac{y}{2^n})\| \\
 &= 2^n \|f(x, \frac{y}{2^n}) - F_1(x, \frac{y}{2^n})\| + 2^n \|f(x, 0) - F_1(x, 0)\| \\
 &+ 2^n \|f(x, \frac{y}{2^n}) - f(x, 0) - F_2(x, \frac{y}{2^n})\| \\
 &\leq (\frac{2}{2^{q_1}})^n \left(\frac{\|x\|^{p_1}}{2^{p_1} - 2} + \frac{\|x\|^{p_2}}{2^{p_2} - 2} \right) \|y\|^{q_1} + \|x\|^{p_3} \frac{2^{q_2}}{2^{q_2} - 2} (\frac{2}{2^{q_2}})^n \|y\|^{q_2}
 \end{aligned}$$

holds for all $n \in N$ and for all $x, y \in X$. Taking $n \rightarrow \infty$ and using $1 < q_1, q_2$, we have (2.13). $\square$

THEOREM 2.3. *Let $p_1, p_2, p_3, q_1, q_2, q_3, \delta_1, \delta_3$ be fixed positive real numbers with $p_1, p_2, p_3 < 1$ and $1 < q_1, q_2, q_3$. Let $f : X \times X \rightarrow Y$ be a mapping such that*

$$\begin{aligned}
 \|D_1 f(x, y, z)\| &\leq (\|x\|^{p_1} + \|y\|^{p_2} + \delta_1) \|z\|^{q_1}, \\
 \|D_2 f(x, y, z)\| &\leq (\|x\|^{p_3} + \delta_3) (\|y\|^{q_2} + \|z\|^{q_3})
 \end{aligned}$$

for all $x, y, z \in X$. Then there exist a unique Cauchy-Jensen mapping $F_1 : X \times X \rightarrow Y$ and a unique biadditive mapping $F_2 : X \times X \rightarrow Y$ satisfying

$$(2.17) \quad \|f(x, y) - F_1(x, y)\| \leq \left(\frac{\|x\|^{p_1}}{2 - 2^{p_1}} + \frac{\|x\|^{p_2}}{2 - 2^{p_2}} + \delta_1 \right) \|y\|^{q_1},$$

$$(2.18) \quad \|f(x, y) - f(x, 0) - F_2(x, y)\| \leq (\|x\|^{p_3} + \delta_3) \frac{2^{q_2}}{2^{q_2} - 2} \|y\|^{q_2},$$

$$(2.19) \quad F_1(x, y) - f(x, 0) = F_2(x, y)$$

for all $x, y \in X$. The mappings $F_1, F_2 : X \times X \rightarrow Y$ are given by (2.20)

$$F_1(x, y) = \lim_{j \rightarrow \infty} \frac{1}{2^j} f(2^j x, y), \quad F_2(x, y) = \lim_{j \rightarrow \infty} 2^j (f(x, \frac{y}{2^j}) - f(x, 0))$$

for all $x, y \in X$.

Proof. By the similar method in the proof of Theorem 2.1 and Theorem 2.2, one obtains that there exist a unique Cauchy-Jensen mapping $F_1 : X \times X \rightarrow Y$ and a unique biadditive mapping $F_2 : X \times X \rightarrow Y$ satisfying (2.17) and (2.18), the mappings $F_1, F_2 : X \times X \rightarrow Y$ are given by (2.20), $f(x, 0) = F_1(x, 0)$ and the equalities in (2.16) hold for all $n \in N$

and for all $x, y \in X$. Hence, the inequality

$$\begin{aligned}
& \|F_1(x, y) - F_1(x, 0) - F_2(x, y)\| \\
&= \|2^n F_1(x, \frac{y}{2^n}) - 2^n F_1(x, 0) - 2^n F_2(x, \frac{y}{2^n})\| \\
&= 2^n \|f(x, \frac{y}{2^n}) - F_1(x, \frac{y}{2^n})\| + 2^n \|f(x, 0) - F_1(x, 0)\| \\
&+ 2^n \|f(x, \frac{y}{2^n}) - f(x, 0) - F_2(x, \frac{y}{2^n})\| \\
&\leq \left(\frac{2}{2^{q_1}}\right)^n \left(\frac{\|x\|^{p_1}}{2 - 2^{p_1}} + \frac{\|x\|^{p_2}}{2 - 2^{p_2}} + \delta_1\right) \|y\|^{q_1} \\
&+ (\|x\|^{p_3} + \delta_3) \frac{2^{q_2}}{2^{q_2} - 2} \left(\frac{2}{2^{q_2}}\right)^n \|y\|^{q_2}
\end{aligned}$$

holds for all $n \in N$ and for all $x, y \in X$. Taking $n \rightarrow \infty$ and using $1 < q_1, q_2$, we have (2.19). $\square$

THEOREM 2.4. *Let $p_1, p_2, p_3, q_1, q_2, q_3, \delta_2, \delta_4$ be fixed positive real numbers with $1 < p_1, p_2, p_3$ and $q_1, q_2, q_3 < 1$. Let $f : X \times X \rightarrow Y$ be a mapping such that*

$$\begin{aligned}
\|D_1 f(x, y, z)\| &\leq (\|x\|^{p_1} + \|y\|^{p_2})(\|z\|^{q_1} + \delta_2), \\
\|D_2 f(x, y, z)\| &\leq \|x\|^{p_3}(\|y\|^{q_2} + \|z\|^{q_3} + \delta_4)
\end{aligned}$$

for all $x, y, z \in X$. Then there exist a unique Cauchy-Jensen mapping $F_1 : X \times X \rightarrow Y$ and a unique biadditive mapping $F_2 : X \times X \rightarrow Y$ such that Let $f : X \times X \rightarrow Y$ be a mapping such that

$$(2.21) \quad \|f(x, y) - F_1(x, y)\| \leq \left(\frac{\|x\|^{p_1}}{2^{p_1} - 2} + \frac{\|x\|^{p_2}}{2^{p_2} - 2}\right)(\|y\|^{q_1} + \delta_2),$$

$$(2.22) \quad \|f(x, y) - f(x, 0) - F_2(x, y)\| \leq \|x\|^{p_3} \left(\frac{2^{q_2}}{2 - 2^{q_2}} \|y\|^{q_2} + \delta_4\right),$$

$$(2.23) \quad F_1(x, y) - F_1(x, 0) = F_2(x, y)$$

for all $x, y \in X$. The mappings $F_1, F_2 : X \times X \rightarrow Y$ are given by

$$(2.24) \quad F_1(x, y) = \lim_{j \rightarrow \infty} 2^j f\left(\frac{x}{2^j}, y\right), \quad F_2(x, y) = \lim_{j \rightarrow \infty} \frac{1}{2^j} f(x, 2^j y)$$

for all $x, y \in X$.

Proof. By the similar method in the proof of Theorem 2.1 and Theorem 2.2, one obtains that there exist a unique Cauchy-Jensen mapping $F_1 : X \times X \rightarrow Y$ and a unique biadditive mapping $F_2 : X \times X \rightarrow Y$ satisfying (2.21) and (2.22), the mappings $F_1, F_2 : X \times X \rightarrow Y$ are given by

(2.24) and the equalities in (2.8) hold for all $n \in N$ and for all $x, y \in X$. Hence, the inequality

$$\begin{aligned}
 & \|F_1(x, y) - F_1(x, 0) - F_2(x, y)\| \\
 &= \left\| \frac{1}{2^n} F_1(x, 2^n y) - \frac{1}{2^n} F_1(x, 0) - \frac{1}{2^n} F_2(x, 2^n y) \right\| \\
 &= \left\| \frac{1}{2^n} f(x, 2^n y) - \frac{1}{2^n} F_1(x, 2^n y) \right\| + \frac{1}{2^n} \|F_1(x, 0) - f(x, 0)\| \\
 &+ \frac{1}{2^n} \|f(x, 2^n y) - f(x, 2^n 0) - F_2(x, 2^n y)\| \\
 &\leq \left(\frac{\|x\|^{p_1}}{2^{p_1} - 2} + \frac{\|x\|^{p_2}}{2^{p_2} - 2} \right) \left(\left(\frac{2^{q_1}}{2} \right)^n \|y\|^{q_1} + \frac{2}{2^n} \delta_2 \right) \\
 &+ \|x\|^{p_3} \left(\left(\frac{2^{q_1}}{2} \right)^n \frac{2^{q_2}}{2 - 2^{q_2}} \|y\|^{q_2} + \frac{1}{2^n} \delta_4 \right).
 \end{aligned}$$

Taking $n \rightarrow \infty$ and using $q_1, q_2 < 1$, we have (2.23). $\square$

COROLLARY 2.5. *Let $0 < p_1, p_2 < 1$ and $0 < q_1, q_2 < 1$ or $1 < q_1, q_2$. Let $f : X \times X \rightarrow Y$ be a mapping such that*

$$(2.25) \quad \|Df(x, y, z, w)\| \leq (\|x\|^{p_1} + \|y\|^{p_2} + \delta_1)(\|z\|^{q_1} + \|w\|^{q_2} + \delta_2)$$

for all $x, y, z, w \in X$, where $\delta_2 = 0$ for $q_1, q_2 > 1$. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \rightarrow Y$ such that

$$\|f(x, y) - F(x, y)\| \leq \left(\frac{\|x\|^{p_1}}{2 - 2^{p_1}} + \frac{\|x\|^{p_2}}{2 - 2^{p_2}} + \delta_1 \right) \frac{1}{2} (\|y\|^{q_1} + \|y\|^{q_2} + \delta_2)$$

for all $x, y \in X$ where $\delta_2 = 0$ for $q_1, q_2 > 1$. The mapping $F : X \times X \rightarrow Y$ is given by

$$F(x, y) = \lim_{j \rightarrow \infty} \frac{1}{2^j} f(2^j x, y)$$

for all $x, y \in X$.

Proof. From (2.25), we know that

$$\begin{aligned}
 \|D_1 f(x, y, z)\| &= \left\| \frac{1}{2} Df(x, y, z, z) \right\| \\
 &\leq \frac{1}{2} (\|x\|^{p_1} + \|y\|^{p_2} + \delta_1) (\|z\|^{q_1} + \|z\|^{q_2} + \delta_2), \\
 \|D_2 f(x, y, z)\| &= \left\| Df\left(\frac{x}{2}, \frac{x}{2}, y, z\right) - \frac{1}{2} Df\left(\frac{x}{2}, \frac{x}{2}, y, y\right) - \frac{1}{2} Df\left(\frac{x}{2}, \frac{x}{2}, z, z\right) \right\| \\
 &\leq \left(\left\| \frac{x}{2} \right\|^{p_1} + \left\| \frac{x}{2} \right\|^{p_2} + \delta_1 \right) \left(\frac{3}{2} \|y\|^{q_1} \right. \\
 &\quad \left. + \frac{3}{2} \|z\|^{q_2} + 2\delta_2 + \frac{1}{2} \|y\|^{q_2} + \frac{1}{2} \|z\|^{q_1} \right)
 \end{aligned}$$

for all $x, y \in X$. Then we can apply the similar method in the proof of Theorem 2.1 for the case $0 < q_1, q_2 < 1$ and apply the similar method in the proof of Theorem 2.3 for the case $1 < q_1, q_2$, and therefore, we get the results in this corollary. $\square$

COROLLARY 2.6. *Let $1 < p_1, p_2$ and $0 < q_1, q_2 < 1$ or $1 < q_1, q_2$. Let $f : X \times X \rightarrow Y$ be a mapping such that*

$$\|Df(x, y, z, w)\| \leq (\|x\|^{p_1} + \|y\|^{p_2})(\|z\|^{q_1} + \|w\|^{q_2} + \delta_2)$$

for all $x, y, z, w \in X$, where $\delta_2 = 0$ for $q_1, q_2 > 1$. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \rightarrow Y$ such that

$$\|f(x, y) - F(x, y)\| \leq \left(\frac{\|x\|^{p_1}}{2^{p_1} - 2} + \frac{\|x\|^{p_2}}{2^{p_2} - 2} \right) \frac{1}{2} (\|y\|^{q_1} + \|y\|^{q_2} + \delta_2)$$

for all $x, y \in X$ where $\delta_2 = 0$ for $q_1, q_2 > 1$. The mapping $F : X \times X \rightarrow Y$ is given by

$$F(x, y) = \lim_{j \rightarrow \infty} 2^j f\left(\frac{x}{2^j}, y\right)$$

for all $x, y \in X$.

Proof. We can use the same method in the proof of Corollary 2.5 and we get the results in this corollary by applying Theorem 2.2 and Theorem 2.4. $\square$

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