

## ON THE STABILITY OF AN $n$ -DIMENSIONAL QUADRATIC EQUATION

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ABSTRACT. Let  $X$  and  $Y$  be vector spaces. In this paper we prove that a mapping  $f : X \rightarrow Y$  satisfies the following functional equation

$$\sum_{1 \leq k < l \leq n} (f(x_k + x_l) + f(x_k - x_l)) - 2(n-1) \sum_{i=1}^n f(x_i) = 0$$

if and only if the mapping  $f$  is quadratic. In addition we investigate the generalized Hyers-Ulam-Rassias stability problem for the functional equation.

### 1. Introduction

The quadratic function  $f(x) = cx^2$  ( $x \in \mathbb{R}$ ), where  $c$  is a real constant, clearly satisfies the equation

$$(1.1) \quad f(x_1 + x_2) + f(x_1 - x_2) = 2(f(x_1) + f(x_2)).$$

Hence, the equation (1.1) is called the quadratic functional equation. In particular, every solution of the quadratic functional equation (1.1) is said to be a quadratic function. It is well known that a function  $f : X \rightarrow Y$  between real vector spaces is a quadratic function (1.1) if and only if there exists a unique symmetric biadditive function  $B : X^2 \rightarrow Y$  such that  $f(x) = B(x, x)$  for all  $x \in X$  (see [1, 6]). The following problem of this kind had been formulated by Ulam during a talk before a Mathematical Colloquium at the University of Wisconsin, Madison, in 1940. Given a metric group  $G(\cdot, d)$ , a number  $\delta > 0$  and a mapping  $h : G \rightarrow G$  which satisfies the inequality  $d(h(x \cdot y), h(x) \cdot h(y)) < \delta$  for all

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$x, y \in G$ , does there exist an automorphism  $g$  of  $G$  and a constant  $k > 0$ , depending only on  $G$ , such that  $d(g(x), h(x)) \leq k\delta$  for all  $x \in G$ ? If the answer is affirmative, we would call the equation  $g(x \cdot y) = g(x) \cdot g(y)$  of automorphism stable. The Hyers-Ulam stability of the quadratic functional equation was first proved by F. Skof for a function  $f : X \rightarrow Y$ , where  $X$  is a normed space and  $Y$  is a Banach space [11]. P.W. Cholewa demonstrated that the theorem of Skof is also valid if  $X$  is replaced by an abelian group [2]. Later, the Hyers-Ulam-Rassias stability of the quadratic functional equation was proved by S. Czerwik [3], J.M. Rassias [8] and Th.M. Rassias ([9, 10]). The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem [5, 7]. In this paper, we will extend Eq.(1.1) to an  $n$ -dimensional quadratic functional equation and then investigate the generalized Hyers-Ulam-Rassias stability of the  $n$ -dimensional quadratic functional equation as follows:

$$(1.2) \quad \sum_{1 \leq k < l \leq n} (f(x_k + x_l) + f(x_k - x_l)) - 2(n-1) \sum_{i=1}^n f(x_i) = 0$$

for all  $x_1, \dots, x_n \in X$ , where  $n \geq 2$  is an integer number.

## 2. Solution of the functional equation (1.2)

**THEOREM 2.1.** *Let  $X$  and  $Y$  be vector spaces. A mapping  $f : X \rightarrow Y$  satisfies the functional equation (1.2) if and only if  $f$  is quadratic.*

*Proof.* Let  $f$  be a quadratic function. Assume the equation (1.2) is true for  $n$  by induction argument. By (1.1),

$$(2.1) \quad f(x_i + x_{n+1}) + f(x_i - x_{n+1}) - 2f(x_i) - 2f(x_{n+1}) = 0$$

for all  $i = 1, \dots, n$ . Adding up (1.2) and (2.1), we have the desired equation (1.2) for  $n+1$ . Conversely, let  $f$  satisfy the equation (1.2). By letting  $x_i = 0$  for all  $i = 1, 2, \dots, n$ , we have  $f(0) = 0$ . Replacing  $x_i = 0$  for all  $i = 3, 4, \dots, n$ , we obtain the equation

$$f(x_1 + x_2) + f(x_1 - x_2) = 2(f(x_1) + f(x_2)),$$

which implies that  $f$  is quadratic. The proof is complete.  $\square$

In Section 3, we investigate the generalized Hyers-Ulam stability of the functional equation (1.2) in the spirit of P. Găvruta [4].

### 3. Stability of the quadratic equation (1.2)

In this section, let  $X$  be a vector space and  $Y$  a Banach space. For the given mapping  $f : X \rightarrow Y$ , we define

$$Df(x_1, \dots, x_n) := \sum_{1 \leq k < l \leq n} (f(x_k + x_l) + f(x_k - x_l)) - 2(n-1) \sum_{i=1}^n f(x_i)$$

for all  $x_1, \dots, x_n \in X$ , where  $n \geq 2$  is an integer number. We denote by  $\varphi : X^n \rightarrow [0, \infty)$  a function such that

$$(3.1) \quad \Phi(x_1, x_2, \dots, x_n) := \sum_{k=0}^{\infty} \left( \frac{1}{4^k} \right) \varphi(2^k x_1, \dots, 2^k x_n) < \infty.$$

LEMMA 3.1. *Let a function  $f : X \rightarrow Y$  satisfy  $f(0) = 0$  and the inequality*

$$(3.2) \quad \left\| \sum_{1 \leq k < l \leq n} (f(x_k + x_l) + f(x_k - x_l)) - 2(n-1) \sum_{i=1}^n f(x_i) \right\| \leq \varphi(x_1, x_2, \dots, x_n)$$

for all  $x_1, \dots, x_n \in X$ , where  $n \geq 2$  is an integer number. Then

$$(3.3) \quad \left\| \frac{f(2^m x)}{4^m} - f(x) \right\| \leq \frac{1}{2n(n-1)} \sum_{k=0}^{m-1} \left( \frac{1}{4^k} \right) \underbrace{\varphi(2^k x, \dots, 2^k x)}_{n\text{-times}}$$

for all  $m \in \mathbb{N}$  and  $x \in X$ .

*Proof.* Now, we are going to prove our assertion by induction on  $m \in \mathbb{N}$ . Put  $x = x_1 = \dots = x_n$  in (3.2). Then we obtain

$$\left\| \frac{f(2x)}{4} - f(x) \right\| \leq \frac{1}{2n(n-1)} \varphi(x, \dots, x).$$

Thus it holds good for  $m = 1$ . We assume that the assertion is true for  $m$ . By replacing  $x$  by  $2^m x$  in the above relation and dividing  $4^m$  the resulting inequality, then we have

$$\left\| \frac{f(2^{m+1}x)}{4^{m+1}} - \frac{f(2^m x)}{4^m} \right\| \leq \frac{1}{4^m} \frac{1}{2n(n-1)} \varphi(2^m x, \dots, 2^m x).$$

And so

$$\begin{aligned}
& \left\| \frac{f(2^{m+1}x)}{4^{m+1}} - f(x) \right\| \\
& \leq \left\| \frac{f(2^{m+1}x)}{4^{m+1}} - \frac{f(2^m x)}{4^m} \right\| + \left\| \frac{f(2^m x)}{4^m} - f(x) \right\| \\
& \leq \frac{1}{2n(n-1)} \sum_{k=0}^m \left( \frac{1}{4^k} \right) \varphi(2^k x, \dots, 2^k x).
\end{aligned}$$

This completes the proof of the lemma.  $\square$

**THEOREM 3.2.** *Assume that a function  $f : X \rightarrow Y$  satisfies  $f(0) = 0$  and the inequality (3.2) for all  $x_1, x_2, \dots, x_n \in X$ . Then there exists a unique quadratic function  $Q : X \rightarrow Y$  satisfying*

$$(3.4) \quad \|f(x) - Q(x)\| \leq \frac{1}{2n(n-1)} \underbrace{\Phi(x, \dots, x)}_{n\text{-times}}$$

for all  $x \in X$ .

*Proof.* In order to prove convergence of the sequence  $\left\{ Q_m(x) = \frac{f(2^m x)}{4^m} \right\}$  we show that  $\{Q_m(x)\}$  is a Cauchy sequence in  $Y$ . By (3.3), we have for  $m_1 > m_2 > 0$ ,

$$\begin{aligned}
(3.5) \quad & \left\| \frac{f(2^{m_1}x)}{4^{m_1}} - \frac{f(2^{m_2}x)}{4^{m_2}} \right\| \\
& = \frac{1}{4^{m_2}} \left\| \frac{f(2^{m_1-m_2} \cdot 2^{m_2}x)}{4^{m_1-m_2}} - f(2^{m_2}x) \right\| \\
& \leq \frac{1}{2n(n-1)} \sum_{k=0}^{m_1-m_2-1} \left( \frac{1}{4^{k+m_2}} \right) \varphi(2^{k+m_2}x, \dots, 2^{k+m_2}x).
\end{aligned}$$

Since the right-hand side of the inequality tends to 0 as  $m_2 \rightarrow \infty$ , the sequence  $\{Q_m(x)\}$  is Cauchy in the Banach space  $Y$ . Therefore we may define a function  $Q : X \rightarrow Y$  by

$$Q(x) = \lim_{m \rightarrow \infty} \frac{f(2^m x)}{4^m}$$

for all  $x \in X$ . By letting  $m \rightarrow \infty$  in (3.3), we arrive at the formula (3.4).

Now we show that  $Q$  satisfies the functional equation (1.2) for all  $x_1, \dots, x_n \in X$ . By definition of  $Q$ ,

$$\begin{aligned} & \left\| \sum_{1 \leq k < l \leq n} (Q(x_k + x_l) + Q(x_k - x_l)) - 2(n-1) \sum_{i=1}^n Q(x_i) \right\| \\ &= \lim_{m \rightarrow \infty} 4^{-m} \left\| \sum_{1 \leq k < l \leq n} (f(2^m(x_k + x_l)) + f(2^m(x_k - x_l))) \right. \\ & \quad \left. - 2(n-1) \sum_{i=1}^n f(2^m x_i) \right\| \\ &\leq \lim_{m \rightarrow \infty} 4^{-m} \varphi(2^m x_1, \dots, 2^m x_n) = 0. \end{aligned}$$

Thus Theorem 2.1 implies that  $Q$  is quadratic. It only remains to claim that  $Q$  is unique. Let  $Q' : X \rightarrow Y$  be another quadratic function which satisfies the inequality (3.4). Since  $Q$  and  $Q'$  are quadratic function, we can easily show that

$$(3.6) \quad Q(2^m x) = 4^m Q(x) \quad \text{and} \quad Q'(2^m x) = 4^m Q'(x)$$

for any  $m \in \mathbb{N}$ . Thus, it follows from (3.4) that

$$\begin{aligned} \|Q(x) - Q'(x)\| &\leq \frac{1}{4^m} (\|Q(2^m x) - f(2^m x)\| + \|f(2^m x) - Q'(2^m x)\|) \\ &\leq \frac{2}{4^m} \frac{1}{2n(n-1)} \Phi(2^m x, \dots, 2^m x). \end{aligned}$$

By letting  $m \rightarrow \infty$ , then we get that  $Q(x) = Q'(x)$  for all  $x \in X$ , which completes the proof of the theorem.  $\square$

**COROLLARY 3.3.** *If a function  $f : X \rightarrow Y$  satisfies  $f(0) = 0$  and the inequality*

$$(3.7) \quad \|Df(x_1, \dots, x_n)\| \leq \varepsilon (\|x_1\|^p + \dots + \|x_n\|^p)$$

*for some  $0 < p < 2$  and for all  $x_1, \dots, x_n \in X$ , then there exists a unique quadratic function  $Q : X \rightarrow Y$  such that*

$$\|f(x) - Q(x)\| \leq \frac{2^{2-p}}{2(n-1)(2^{2-p} - 1)} \varepsilon \|x\|^p$$

*for all  $x \in X$ .*

**COROLLARY 3.4.** *If a function  $f : X \rightarrow Y$  satisfies  $f(0) = 0$  and the inequality*

$$\|Df(x_1, \dots, x_n)\| \leq \varepsilon$$

for all  $x_1, \dots, x_n \in X$ , then there exists a unique quadratic function  $Q : X \rightarrow Y$  such that

$$\|f(x) - Q(x)\| \leq \frac{2\varepsilon}{3n(n-1)}$$

for all  $x \in X$ .

Now, we investigate another stability question controlled by a function  $\varphi : X^n \rightarrow [0, \infty)$ .

**THEOREM 3.5.** Assume that a function  $f : X \rightarrow Y$  satisfies the inequality (3.2) for all  $x_1, x_2, \dots, x_n \in X$  and  $\varphi$  satisfies the condition

$$\hat{\Phi}(x_1, x_2, \dots, x_n) := \sum_{k=0}^{\infty} 4^k \varphi\left(\frac{x_1}{2^k}, \dots, \frac{x_n}{2^k}\right) < \infty.$$

Then there exists a unique quadratic function  $Q : X \rightarrow Y$  satisfying

$$\|f(x) - Q(x)\| \leq \frac{1}{2n(n-1)} \underbrace{\hat{\Phi}(x, \dots, x)}_{n\text{-times}}.$$

*Proof.* The proof is similar to that of Theorem 3.2.  $\square$

**COROLLARY 3.6.** If a function  $f : X \rightarrow Y$  satisfies the inequality (3.7) for some  $p > 2$  and for all  $x_1, \dots, x_n \in X$ , then there exists a unique quadratic function  $Q : X \rightarrow Y$  such that

$$\|f(x) - Q(x)\| \leq \frac{2^{p-2}}{2(n-1)(2^{p-2}-1)} \varepsilon \|x\|^p$$

for all  $x \in X$ .

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