

ON STRONG C-INTEGRAL OF BANACH-VALUED FUNCTIONS

DAFANG ZHAO* AND GUOJU YE**

ABSTRACT. In this paper, we define and study the Banach-valued C-integral and strong C-integral, We prove that the C-integral and the strong C-integral are equivalent if and only if the Banach space is finite dimensional. We also study the primitive of the strong C-integral in terms of the C-variational measures.

1. Introduction

In 1996 B.Bongiorno introduced a constructive integration process of Riemann type, called C-integral, which is the smallest integral that includes the Lebesgue integral and derivatives. Some properties of the C-integral of real-valued functions were studied in [1, 2, 7].

In this paper, we define and study the C-integral and the strong C-integral of functions mapping an interval $[a, b]$ into a Banach space X . We prove that the C-integral and the strong C-integral are equivalent if and only if the Banach space is finite dimensional. If the function $F : \mathcal{I} \rightarrow X$ is differentiable almost everywhere on $[a, b]$, then it is the indefinite strong C-integral of f if and only if the C-variational measure V_*F is absolutely continuous. Consequently, we prove that every function of bounded variation is a multiplier for the strong C-integral.

2. Definitions and basic properties

Throughout this paper, $[a, b]$ is a compact interval in $\mathbb{R}$. X will denote a real Banach space with norm $\|\cdot\|$ and its dual X^* . $\mathcal{I}$ denote the family

Received November 2, 2006.

2000 Mathematics Subject Classification: Primary 28B05, 26A39.

Key words and phrases: C-integral, strong C-integral, C-variational measure.

Supported by NNSF of China (10571085) and by Science Foundation of Hohai University.

of all subintervals of $[a, b]$. $\overline{co}(Y)$ denote the closed convex hull of the set Y if $Y \subset X$.

A partition D is a finite collection of interval-point pairs $\{([u_i, v_i], \xi_i)\}_{i=1}^n$, where $\{[u_i, v_i]\}_{i=1}^n$ are non-overlapping subintervals of $[a, b]$. $\delta(\xi)$ is a positive function on $[a, b]$, i.e. $\delta(\xi) : [a, b] \rightarrow \mathbb{R}^+$. We say that $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ is

- (1) a partial partition of $[a, b]$ if $\bigcup_{i=1}^n [u_i, v_i] \subset [a, b]$,
- (2) a partition of $[a, b]$ if $\bigcup_{i=1}^n [u_i, v_i] = [a, b]$,
- (3) δ - fine *McShane partition* of $[a, b]$ if $[u_i, v_i] \subset B(\xi_i, \delta(\xi_i)) = (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ and $\xi_i \in [a, b]$ for all $i=1, 2, \dots, n$,
- (4) δ - fine *C-partition* of $[a, b]$ if it is a δ - fine McShane partition of $[a, b]$ and satisfying the condition

$$\sum_{i=1}^n \text{dist}(\xi_i, [u_i, v_i]) < \frac{1}{\varepsilon},$$

here $\text{dist}(\xi_i, [u_i, v_i]) = \inf\{|t_i - \xi_i| : t_i \in [u_i, v_i]\}$.

Given a δ - fine *C-partition* $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ we write

$$S(f, D) = \sum_{i=1}^n f(\xi_i)(v_i - u_i)$$

for integral sums over D , whenever $f : [a, b] \rightarrow X$.

DEFINITION 2.1. A function $f : [a, b] \rightarrow X$ is C-integrable if there exists a vector $A \in X$ such that for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : [a, b] \rightarrow \mathbb{R}^+$ such that

$$\|S(f, D) - A\| < \varepsilon$$

for each δ - fine *C-partition* $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of $[a, b]$. A is called the *C-integral* of f on $[a, b]$, and we write $A = \int_a^b f$ or $A = (C) \int_a^b f$.

The function f is C-integrable on the set $E \subset [a, b]$ if the function $f\chi_E$ is C-integrable on $[a, b]$. We write $\int_E f = \int_a^b f\chi_E$.

The basic properties of the C-integral, for example, linearity and additivity with respect to intervals can be founded in [7]. We do not present them here. The reader is referred to [7] for the details.

LEMMA 2.2. (Saks-Henstock) Let $f : [a, b] \rightarrow X$ is C-integrable on $[a, b]$. Then for $\varepsilon > 0$ there is a positive function $\delta(\xi) : [a, b] \rightarrow \mathbb{R}^+$ such that

$$\|S(f, D) - \int_a^b f\| < \varepsilon$$

for each δ - fine C-partition $D = \{([u, v], \xi)\}$ of $[a, b]$. Particulary, if $D' = \{([u_i, v_i], \xi_i)\}_{i=1}^m$ is an arbitrary δ - fine partial C-partition of $[a, b]$, we have

$$\|S(f, D') - \sum_{i=1}^m \int_{u_i}^{v_i} f\| \leq \epsilon.$$

Proof. The proof is similar to the case for Banach-valued Henstock integrable functions and the reader is referred to [9, Lemma 3.4.1.] for details. $\square$

THEOREM 2.3. Let $f : [a, b] \rightarrow X$ is C-integrable on $[a, b]$.

- (1) for each $x^* \in X^*$, the function x^*f is C-integrable on $[a, b]$ and $\int_a^b x^*f = x^*(\int_a^b f)$.
- (2) If $T : X \rightarrow Y$ is a continuous linear operator, then Tf is C-integrable on $[a, b]$ and $\int_a^b Tf = T(\int_a^b f)$.

Proof. The proof is too easy and will be omitted. The reader is referred to [10, Theorem 2.9.] for details. $\square$

DEFINITION 2.4. A function $f : [a, b] \rightarrow X$ is strongly C-integrable if there exists an additive function $F : \mathcal{I} \rightarrow X$ such that for each $\epsilon > 0$ there is a positive function $\delta(\xi) : [a, b] \rightarrow R^+$ such that

$$\sum_{i=1}^n \|f(\xi_i)(v_i - u_i) - F(u_i, v_i)\| < \epsilon$$

for each δ - fine C-partition $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of $[a, b]$. We denote $F(u_i, v_i) = F(v_i) - F(u_i)$.

THEOREM 2.5. Let X be a Banach space of finite dimension. $f : [a, b] \rightarrow X$ is C-integrable on $[a, b]$ if and only if f is strongly C-integrable on $[a, b]$.

Proof (Sufficiency). From the definitions of the strong C-integral and C-integral, if f is strongly C-integrable on $[a, b]$, then f is C-integrable on $[a, b]$. $\square$

Proof (Necessity). f is C-integrable on $[a, b]$, then there is a positive function $\delta(\xi) : [a, b] \rightarrow R^+$ such that

$$\|\sum f(\xi)(v - u) - F(u, v)\| < \epsilon$$

for each δ -fine C -partition $D = \{([u, v], \xi)\}$ of $[a, b]$. Let $\{e_1, e_2, \dots, e_n\}$ be a base of X and $g_i : [a, b] \rightarrow R$ ($i = 1, 2, \dots, n$). By the Hahn-Banach Theorem, for each e_i there is $x_i^* \in X^*$ such that

$$(2.1) \quad x_i^*(e_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

for $i, j = 1, 2, \dots, n$ and therefore $x_i^*(f) = \sum_{j=1}^n g_j x_i^*(e_j) = g_i$. Since $g_i : [a, b] \rightarrow R$ is C -integrable on $[a, b]$ from Theorem 2.3, for each $\varepsilon > 0$ there is a positive function $\delta_i(\xi) : [a, b] \rightarrow R^+$ such that

$$|S(g_i, D_i) - \sum \int_u^v g_i| < \epsilon$$

for each δ_i -fine C -partition $D_i = \{([u, v], \xi)\}$ of $[a, b]$. By an easy adaptation of Saks-Henstock Lemma we have

$$\sum |g_i(\xi)(v - u) - \int_u^v g_i| < 2\epsilon.$$

We also have

$$F(u, v) = \int_u^v f = \int_u^v \sum_{i=1}^n g_i e_i = \sum_{i=1}^n \int_u^v g_i e_i = \sum_{i=1}^n e_i G_i(u, v)$$

where $G_i(u, v) = \int_u^v g_i$. Let $\delta(\xi) < \delta_i(\xi)$ for $i = 1, 2, \dots, n$ and consequently

$$\begin{aligned} & \sum \|f(\xi)(v - u) - F(u, v)\| \\ &= \sum \left\| \sum_{i=1}^n g_i(\xi) e_i (v - u) - \sum_{i=1}^n e_i G_i(u, v) \right\| \\ &\leq \sum_{i=1}^n \|e_i\| \sum |g_i(\xi)(v - u) - G_i(u, v)| \\ &< \epsilon \cdot \sum_{i=1}^n \|e_i\| \end{aligned}$$

for each δ -fine C -partition $D = \{([u, v], \xi)\}$ of $[a, b]$. Hence f is strongly C -integrable on $[a, b]$. □

3. the C-variational measure and the strong C-integral

Let $F : [a, b] \rightarrow X$, arbitrary $E \subset [a, b]$ and a positive function $\delta(\xi) : E \rightarrow R^+$, Let us set

$$V(F, \delta, E) = \sup_D \sum_i \|F(u_i, v_i)\|$$

where the supremum is take over all δ - fine *partial C-partition* $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of $[a, b]$ with $\xi_i \in E$. We put

$$V_*F(E) = \liminf_{\delta \rightarrow 0} V(F, \delta, E)$$

where the infimum is take over all function $\delta(\xi) : E \rightarrow R^+$.

It is easy to know that the set function $V_*F(E)$ is a Borel metric outer measure, known as the C-variational measure generated by F .

DEFINITION 3.1. $V_*F(E)$ is said to be absolutely continuous (AC) on a set E if for each set $N \subset E$ such that $V_*F(N) = 0$ whenever $\mu(N) = 0$.

DEFINITION 3.2. A function $F : [a, b] \rightarrow X$ is differentiable at $\xi \in [a, b]$ if there is a $f(\xi) \in X$ such that

$$\lim_{\delta \rightarrow 0} \left\| \frac{F(\xi + \delta) - F(\xi)}{\delta} - f(\xi) \right\| = 0.$$

We denote $f(\xi) = F'(\xi)$ the derivative of F at ξ .

THEOREM 3.3. Let $F : \mathcal{I} \rightarrow X$ be differentiable almost everywhere on $[a, b]$. Then F is the indefinite strong C-integral of f if and only if the C-variational measure V_*F is AC.

Proof (Necessity). Let $E \subset [a, b]$ and $\mu(E) = 0$. Assume $E_n = \{\xi \in E : n - 1 \leq \|f(\xi)\| < n\}$ for $n = 1, 2, \dots$. Then we have $E = \bigcup E_n$ and $\mu(E_n) = 0$, so there are open sets G_n such that $E_n \subset G_n$ and $\mu(G_n) < \frac{\epsilon}{n \cdot 2^n}$.

By the Saks-Henstock Lemma, there exists a positive function δ_0 such that

$$\sum \|f(\xi_i)(v_i - u_i) - F(u_i, v_i)\| < \epsilon$$

for each δ_0 - fine *partial C-partition* $D = \{([u_i, v_i], \xi_i)\}$ of $[a, b]$.

For $\xi \in E_n$, take $\delta_n(\xi) > 0$ such that $B(\xi, \delta_n(\xi)) \subset G_n$. Let

$$\delta(\xi) = \min\{\delta_0(\xi), \delta_n(\xi)\}.$$

Assume $D' = \{([u, v], \xi)\}$ is a δ - fine *partial C-partition* with $\xi \in E$. We have

$$\begin{aligned}
 \sum \|F(u, v)\| &= \sum \|F(u, v) - f(\xi)(v - u) + f(\xi)(v - u)\| \\
 &\leq \sum \|F(u, v) - f(\xi)(v - u)\| + \sum \|f(\xi)(v - u)\| \\
 &< \epsilon + \sum_n \sum_{\xi \in E_n} \|f(\xi)(v - u)\| \\
 &< \epsilon + \sum_n n \frac{\epsilon}{n \cdot 2^n} = 2\epsilon
 \end{aligned}$$

This shows that $V_*F(E) < 2\epsilon$. Hence the C-variational measure V_*F is AC as desired. $\square$

Proof (Sufficiency). There exists a set $E \subset [a, b]$ be of measure zero such that $f(\xi) \neq F'(\xi)$ or $F'(\xi)$ does not exist for $\xi \in E$. We can define a function as follows

$$(3.1) \quad f(x) = \begin{cases} F'(\xi) & \text{if } \xi \in [a, b] \setminus E, \\ \theta & \text{if } \xi \in E. \end{cases}$$

Then for $\xi \in [a, b] \setminus E$, by the definition of derivative, for each $\varepsilon > 0$, there is a positive function $\delta_1(\xi)$ such that

$$\|f(\xi)(v - u) - F(u, v)\| < \frac{\epsilon^2}{1 + \epsilon(b - a)}(\text{dist}(\xi, [u, v]) + v - u)$$

for each interval $[u, v] \subset (\xi - \delta_1(\xi), \xi + \delta_1(\xi))$.

V_*F is AC, then for $\xi \in E$, there is a positive function $\delta_2(\xi)$ such that

$$\sum \|F(u, v)\| < \epsilon$$

for each δ_2 - fine *partial C-partition* $D_0 = \{([u, v], \xi)\}$ with $\xi \in E$.

Define a positive function $\delta(\xi)$ as follows

$$(3.2) \quad \delta(\xi) = \begin{cases} \delta_1(\xi) & \text{if } \xi \in [a, b] \setminus E, \\ \delta_2(\xi) & \text{if } \xi \in E. \end{cases}$$

Then for each δ - fine C -partition of $[a, b]$, we have

$$\begin{aligned}
& \sum \|f(\xi)(v - u) - F(u, v)\| \\
&= \sum_{\xi \in E} \|F(u, v) - f(\xi)(v - u)\| + \sum_{\xi \in [a, b] \setminus E} \|F(u, v) - f(\xi)(v - u)\| \\
&\leq \epsilon + \frac{\epsilon^2}{1 + \epsilon(b - a)} \sum_{\xi \in [a, b] \setminus E} (\text{dist}(\xi, [u, v]) + v - u) \\
&< \epsilon + \frac{\epsilon^2}{1 + \epsilon(b - a)} \left(\frac{1}{\epsilon} + b - a\right) = 2\epsilon
\end{aligned}$$

Hence f is strong C-integrable on $[a, b]$ with indefinite strong C-integral F . $\square$

DEFINITION 3.4. Let $G : [a, b] \rightarrow R$. A function $F : [a, b] \rightarrow X$ is C-Stieltjes integrable with respect to G on $[a, b]$ if there exists a vector $A \in X$ such that for each $\epsilon > 0$ there is a positive function $\delta(\xi) : [a, b] \rightarrow R^+$ such that

$$\|S(F, G, D) - A\| < \epsilon$$

for each δ - fine C -partition $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of $[a, b]$, whenever

$$S(F, G, D) = \sum_{i=1}^n F(\xi_i)(G(v_i) - G(u_i)).$$

A is called the C -Stieltjes integral of F with respect to G on $[a, b]$, and we write $A = (CS) \int_a^b F dG$.

Similar to [8, Proposition 5.], we have the following Lemma.

LEMMA 3.5. Let $G : [a, b] \rightarrow R$ be a non decreasing function. If a function $F : [a, b] \rightarrow X$ is C-Stieltjes integrable with respect to G , then for each $[u, v] \in [a, b]$, we have

$$(CS) \int_u^v F dG \in \overline{\text{co}}(\{G(u, v)x : x \in X \text{ and } x = F(\xi) \text{ for some } \xi \in [u, v]\})$$

THEOREM 3.6. Let $f : [a, b] \rightarrow X$ be strongly C-integrable on $[a, b]$ and $F(x) = \int_a^x f$ for each $x \in [a, b]$. If $G : [a, b] \rightarrow R$ is a function of bounded variation, then Gf is strongly C-integrable and

$$\int_a^b Gf = G(b)F(b) - (CS) \int_a^b F dG.$$

Proof. Let $\epsilon > 0$, arbitrary $E \subset [a, b]$ with $\mu(E) = 0$. Assume $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ is an arbitrary δ -fine *partial C-partition* with $\xi_i \in E$.

It is easy to know that F is continuous on $[a, b]$. G is of bounded variation, then the C-Stieltjes integral $(CS) \int_a^b F dG$ exists on $[a, b]$. We can assume G is non decreasing and with upper bounded $M > 0$ on $[a, b]$, then for each i , there are $x_1^{(i)}, x_2^{(i)}, \dots, x_{m_i}^{(i)} \in [u_i, v_i]$ and numbers $\lambda_1^{(i)}, \lambda_2^{(i)}, \dots, \lambda_{m_i}^{(i)}$ with $\sum_{j=1}^{m_i} \lambda_j^{(i)} = 1$ such that

$$\left\| \sum_{j=1}^{m_i} \lambda_j^{(i)} G(u_i, v_i) F(x_j^{(i)}) - \int_{u_i}^{v_i} F dG \right\| \leq \frac{\epsilon G(u_i, v_i)}{nV(G, [a, b])}$$

where $V(G, [a, b])$ denote the variation of G over the interval $[a, b]$.

We define a function by

$$\int_a^x Gf = H(x) = G(x)F(x) - (CS) \int_a^x F dG$$

and consequently have

$$\begin{aligned} & \|H(v_i) - H(u_i)\| \\ = & \|G(v_i)F(v_i) - G(u_i)F(u_i) - (CS) \int_{u_i}^{v_i} F dG\| \\ = & \|G(v_i)[F(v_i) - F(u_i)] + (G(v_i) - G(u_i))[F(u_i) - \sum_{j=1}^{m_i} \lambda_j^{(i)} F(x_j^{(i)})] + \\ & (G(v_i) - G(u_i)) \sum_{j=1}^{m_i} \lambda_j^{(i)} F(x_j^{(i)}) - (CS) \int_{u_i}^{v_i} F dG\| \\ \leq & |G(v_i)| \cdot \|F(u_i, v_i)\| + G(u_i, v_i) \|F(u_i) - \sum_{j=1}^{m_i} \lambda_j^{(i)} F(x_j^{(i)})\| + \\ & \|G(u_i, v_i) \sum_{j=1}^{m_i} \lambda_j^{(i)} F(x_j^{(i)}) - (CS) \int_{u_i}^{v_i} F dG\| \\ \leq & M \|F(u_i, v_i)\| + G(u_i, v_i) \left\| \sum_{j=1}^{m_i} \lambda_j^{(i)} [F(u_i) - F(x_j^{(i)})] \right\| + \frac{\epsilon G(u_i, v_i)}{nV(G, [a, b])} \end{aligned}$$

$$\begin{aligned}
&\leq M\|F(u_i, v_i)\| + V(G, [a, b]) \sum_{j=1}^{m_i} \lambda_j^{(i)} \|F(u_i) - F(x_j^{(i)})\| + \frac{\epsilon G(u_i, v_i)}{nV(G, [a, b])} \\
&\leq M\|F(u_i, v_i)\| + V(G, [a, b]) \sum_{j=1}^{m_i} \lambda_j^{(i)} \|F(u_i) - F(x_{N(i)}^{(i)})\| + \frac{\epsilon G(u_i, v_i)}{nV(G, [a, b])} \\
&= M\|F(u_i, v_i)\| + V(G, [a, b]) \|F(u_i) - F(x_{N(i)}^{(i)})\| + \frac{\epsilon G(u_i, v_i)}{nV(G, [a, b])}
\end{aligned}$$

where

$$\|F(u_i) - F(x_{N(i)}^{(i)})\| = \max\{\|F(u_i) - F(x_j^{(i)})\|\}$$

for each $j \in \{1, 2, \dots, m_i^{(i)}\}$. V_*F is AC from Theorem 3.3, then there exists a positive function $\delta(\xi)$ such that

$$\sum_{i=1}^n \|F(u_i, v_i)\| < \frac{\epsilon}{M + V(G, [a, b])}.$$

Therefore

$$\begin{aligned}
\sum_{i=1}^n \|H(v_i) - H(u_i)\| &\leq M \sum_{i=1}^n \|F(u_i, v_i)\| + \frac{\epsilon \sum_{i=1}^n G(u_i, v_i)}{nV(G, [a, b])} + \\
&\quad V(G, [a, b]) \sum_{i=1}^n \|F(u_i) - F(x_{N(i)}^{(i)})\| \\
&\leq (M + V(G, [a, b])) \frac{\epsilon}{M + V(G, [a, b])} + \epsilon = 2\epsilon
\end{aligned}$$

and it follows that V_*H is AC. We also have that $H(x)$ is differentiable almost everywhere and $H'(x) = G(x)f(x)$ a.e. on $[a, b]$, then Gf is strongly C-integrable on $[a, b]$ from Theorem 3.3. $\square$

Consequently, we can easily get the following theorem.

THEOREM 3.7. *Let $f : [a, b] \rightarrow X$ and $G : [a, b] \rightarrow R$. If Gf is strongly C-integrable on $[a, b]$ for every strongly C-integrable f , then G is equivalent to a function of bounded variation on $[a, b]$.*

Acknowledgement. The authors are grateful to Professor Luisa Di Piazza for her interesting papers[6,7,8].

References

- [1] B. Bongiorno, *On the Minimal Solution of the Problem of Primitives*, J. Math. Anal. Appl. (251) (2000), No.2, 479-487.

- [2] B. Bongiorno, L. Di Piazza, and D. Preiss, *A constructive minimal integral which includes Lebesgue integrable functions and derivatives*, J. London Math. Soc. **(2) 62** (2000), No.1, 117-126.
- [3] A. M. Bruckner, R. J. Fleissner and J. Fordan, *The minimal integral which includes Lebesgue integrable functions and derivatives*, Collq. Mat. **50** (1986), 289-293
- [4] R. A. Gondon, *The integrals of Lebesgue, Denjoy, Perron and Henstock*, Graduate Studies in Mathematics, Vol. **4**, American Math. Soc., Providence, RI, 1994. MR **95m**:26010.
- [5] P. Y. Lee, *Lanzhou Lectures on Henstock Integration*, World Scientific, Singapore, 1989.
- [6] L. Di Piazza, *Variational measures in the theory of the integration in R^m* , Cze. J. Math. **51** (126) (2001), 95-110.
- [7] L. Di Piazza, *A Riemann-type minimal integral for the classical problem of primitives*, Rend. Istit. Mat. Univ. Trieste Vol. XXXIV, (2002), 143-153.
- [8] L. Di Piazza and V. M. arraffa, *The McShane, PU, and Henstock integrals of Banach valued functions*, Cze. J. Math. **52** (127) (2002), 609-633.
- [9] S. Schwabik and Guoju Ye, *Topics in Banach space integration*, World Scientific, (2005).
- [10] Dafang Zhao and Guoju Ye, *On AP-Henstock-Stieltjes integral*, J. Chungcheong Math. Soc. **19** (2) (2006), 177-188.

*

College of Science
Hohai University
Nanjing, 210098, People's Republic of China
E-mail: dafangzhao@hhu.edu.cn

**

College of Science
Hohai University
Nanjing, 210098, People's Republic of China
E-mail: yegj@hhu.edu.cn