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GENERALIZED STABILITY OF EULER-LAGRANGE TYPE QUADRATIC MAPPINGS

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Abstract. In this paper, we investigate the generalized Hyers–Ulam–Rasssias stability of the following Euler-Lagrange type quadratic functional equation $f(ax+by+cz)+f(ax+by-cz)+f(ax-by+cz)+f(ax-by-cz)=4a^2f(x)+4b^2f(y)+4c^2f(z)$.

1. Introduction

In 1940, S. M. Ulam [12] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms. *Let G be a group and let G' be a metric group with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?*

In 1941, D. H. Hyers [3] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon$$

for all $x, y \in E$. It was shown that the limit $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \le \epsilon.$$

Let E_1 and E_2 be real vector spaces. A function $f : E_1 \to E_2$, there exists a quadratic function if and only if f is a solution function of the

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quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y). \tag{1.1}$$

A stability problem for the quadratic functional equation (1.1) was solved by F. Skof [11] for mapping $f: E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space.

In 1978, Th. M. Rassias [7] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.* S. Czerwik [2] proved the Hyers-Ulam-Rassias stability of quadratic functional equation (1.1). Let E_1 and E_2 be a real normed space and a real Banach space, respectively, and let $p \neq 2$ be a positive constant. If a function $f: E_1 \to E_2$ satisfies the inequality

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \leq \epsilon(\|x\|^p+\|y\|^p)$$

for some $\epsilon > 0$ and for all $x, y \in E_1$, then there exists a unique quadratic function $q: E_1 \to E_2$ such that

$$\|f(x)-q(x)\| \leq \frac{2\epsilon}{|4-2^p|}\|x\|^p$$

for all $x \in G$. In partiqular , we note that J.M. Rassias introduced the Euler-Lagrange quadratic mappings, motivated from the following pertinent algebraic equation

$$|ax+by|^2+|bx-ay|^2=(a^2+b^2)[|x|^2+|y|^2]. \tag{1.2}$$

Thus the second author of this paper introduced ad investigated the stability problem of Ulam for the relative Euler-Lagrange functional equation

$$f(ax+by)+f(bx-ay)=(a^2+b^2)[f(x)+f(y)] \tag{1.3}$$

in the publications[8-10].

Recently, S.M Jung [5] and J. Bae, K. Jun and S. Jung [1] have generalized the equation (1.1) to

$$\begin{aligned} f(x+y+z) &+ f(x-y+z)+f(x+y-z)+f(x-y-z) \\ &= 4f(x)+4f(y)+4f(z) \end{aligned} \tag{1.4}$$

and then have investigated the general solution and the stability problem for the functional equation.

Now, we consider the following functional equations

$$\begin{aligned} f(ax+by+cz) &+ f(ax+by-cz) \\ &+ f(ax-by+cz)+f(ax-by-cz) \\ &= 4a^2f(x)+4b^2f(y)+4c^2f(z), \end{aligned} \tag{1.5}$$

where $a, b, c \neq 0$ are real numbers.

In this paper, we will establish the general solution and the generalized Hyers-Ulam-Rassias stability problem for the equation (1.5) in Banach spaces.

2. Euler-Lagrange type quadratic mapping in Banach spaces

LEMMA 2.1. *Let X and Y be vector spaces. If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\begin{aligned} f(ax+by+cz) &+ f(ax+by-cz) \\ &+ f(ax-by+cz) + f(ax-by-cz) \\ &= 4a^2 f(x) + 4b^2 f(y) + 4c^2 f(z) \end{aligned} \tag{2.1}$$

for all $x, y, z \in X$, then the mapping f is quadratic and $f(\lambda^n x) = \lambda^{2n} f(x)$, where $\lambda = a, b$ or c.

Proof. Letting $x = y$ in (2.1), we get

$$\begin{aligned} f((a+b)x+cz) &+ f((a+b)x-cz) \\ &+ f((a-b)x+cz) + f((a-b)x-cz) \\ &= 4a^2 f(x) + 4b^2 f(x) + 4c^2 f(z) \end{aligned} \tag{2.2}$$

for all $x, z \in X$. Setting $y = -x$ in (2.1), we obtain

$$\begin{aligned} f((a-b)x+cz) &+ f((a-b)x-cz) \\ &+ f((a+b)x+cz) + f((a+b)x-cz) \\ &= 4a^2 f(x) + 4b^2 f(-x) + 4c^2 f(z). \end{aligned} \tag{2.3}$$

By (2.2) and (2.3), we conclude that f is even. And by setting $y = 0$ and $z = 0$ in (2.1), we get $f(ax) = a^2 f(x)$ for all $x \in X$. So, it is easy to verify $f(a^n x) = a^{2n} f(x)$ by induction. Similarly, we have the identity for b and c. Now, substituting 0 for z in (2.1), one obtains

$$\begin{aligned} f(ax+by) + f(ax-by) &= 2a^2 f(x) + 2b^2 f(y) \\ &= 2f(ax) + 2f(by). \end{aligned}$$

for all $x, y \in X$. Hence f is quadratic. □

The mapping $f : X \to Y$ given in the statement of Lemma 2.1 is called an *Euler-Lagrange type quadratic mapping*. Putting $z = 0$ in (2.1) with $a = 1 = b$, we get the quadratic mapping $f(x+y)+f(x-y) = 2f(x) + 2f(y)$.

From now on, Let X and Y be a normed vector space and a Banach space, respectively.

For a given mapping $f : X \to Y$, we define

$$\begin{aligned} Df(x,y,z) &:= f(ax+by+cz) + f(ax+by-cz) + f(ax-by+cz) \\ &+ f(ax-by-cz) - 4a^2 f(x) - 4b^2 f(y) - 4c^2 f(z) \end{aligned}$$

for all $x, y, z \in X$

THEOREM 2.2. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^3 \to [0, \infty)$ such that*

$$\Phi(x,y,z) := \sum_{j=1}^{\infty} a^{2j} \phi\Big(\frac{x}{a^j}, \frac{y}{a^j}, \frac{z}{a^j}\Big) < \infty, \tag{2.4}$$

$$\|Df(x,y,z)\| \le \phi(x,y,z) \tag{2.5}$$

for all $x, y, z \in X$. Then there exists a unique Euler-Lagrange type quadratic mapping $Q : X \to Y$ such that $DQ(x,y,z) = 0$ and

$$\|f(x) - Q(x)\| \le \frac{1}{4a^2}\Phi(x,0,0) \tag{2.6}$$

for all $x \in X$.

Proof. Letting $y = 0$ and $z = 0$ in (2.5), we get

$$\|f(ax) - a^2 f(x)\| \le \frac{1}{4}\phi(x,0,0)$$

for all $x \in X$. So

$$\Big\|f(x) - a^2 f(\frac{x}{a})\Big\| \le \frac{1}{4}\phi\Big(\frac{x}{a}, 0, 0\Big)$$

for all $x \in X$. Hence

$$\begin{aligned} \Big\|a^{2l} f(\frac{x}{a^l}) - a^{2m} f(\frac{x}{a^m})\Big\| &\le \sum_{j=l+1}^{m} \Big\|a^{2(j-1)} f(\frac{x}{a^{j-1}}) - a^{2j} f(\frac{x}{a^j})\Big\| \\ &\le \sum_{j=l+1}^{m} \frac{1}{4} a^{2(j-1)} \phi\Big(\frac{x}{a^j}, 0, 0\Big) \end{aligned} \tag{2.7}$$

for all $x \in X$. It means that a sequence $\{a^{2n} f(\frac{x}{a^n})\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{a^{2n} f(\frac{x}{a^n})\}$ converges. So one can define a mapping $Q : X \to Y$ by $Q(x) := \lim_{n\to\infty} a^{2n} f(\frac{x}{a^n})$ for all $x \in X$.

By (2.4) and (2.5),

$$\begin{aligned}\|DQ(x,y,z)\| &= \lim_{n\to\infty} a^{2n}\Big\|Df\Big(\frac{x}{a^n},\frac{y}{a^n},\frac{z}{a^n}\Big)\Big\| \\ &\le \lim_{n\to\infty} a^{2n}\phi\Big(\frac{x}{a^n},\frac{y}{a^n},\frac{z}{a^n}\Big) = 0\end{aligned}$$

for all $x, y, z \in X$. So $DQ(x,y,z) = 0$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic.

Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.7), we get the approximation (2.6) of f by Q.

Now, let $Q' : X \longrightarrow Y$ be another quadratic mapping satisfying (2.6). Then we obtain

$$\begin{aligned}\|Q(x) - Q'(x)\| &= a^{2n}\Big\|Q\big(\frac{x}{a^n}\big) - Q\big(\frac{x}{a^n}\big)\Big\| \\ &\le a^{2n}\Big[\Big\|Q\big(\frac{x}{a^n}\big) - f\big(\frac{x}{a^n}\big)\Big\| + \Big\|Q'\big(\frac{x}{a^n}\big) - f\big(\frac{x}{a^n}\big)\Big\|\Big] \\ &\le \frac{1}{2}a^{2(n-1)}\Phi\Big(\frac{x}{a^n},0,0\Big),\end{aligned}$$

which tends to zero as $n \to \infty$. So we can conclude that $Q(x) = Q'(x)$ for all $x \in X$. This proves the uniqueness of Q. Hence the mapping $Q : X \to Y$ is a unique quadratic mapping satisfying (2.6). □

Corollary 2.3. *Let p and θ be positive real numbers such that either $p > 2$ and $|a| > 1$ or $p < 2$ and $|a| < 1$, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\|Df(x,y,z)\| \le \theta(\|x\|^p + \|y\|^p + \|z\|^p), \tag{2.8}$$

for all $x, y, z \in X$. Then there exists a unique Euler-Lagrange type quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{\theta \cdot \|x\|^p}{4(|a|^p - a^2)} \tag{2.9}$$

for all $x \in X$.

Proof. Define $\phi(x,y,z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$, and apply Theorem 2.2. □

Theorem 2.4. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^3 \to [0,\infty)$ such that*

$$\Phi(x,y,z) := \sum_{j=0}^{\infty}\frac{1}{a^{2j}}\phi(a^jx, a^jy, a^jz) < \infty, \tag{2.10}$$

$$\|Df(x,y,z)\| \le \phi(x,y,z) \tag{2.11}$$

for all $x, y, z \in X$. Then there exists a unique Euler-Lagrange type quadratic mapping $Q : X \to Y$ such that $DQ(x, y, z) = 0$ and

$$\|f(x) - Q(x)\| \leq \frac{1}{4a^2}\Phi(x, 0, 0) \tag{2.12}$$

for all $x \in X$.

Proof. Letting $y = 0$ and $z = 0$ in (2.11), we get

$$\|f(ax) - a^2 f(x)\| \leq \frac{1}{4}\phi(x, 0, 0)$$

for all $x \in X$. So

$$\left\|f(x) - \frac{1}{a^2}f(ax)\right\| \leq \frac{1}{4a^2}\phi(x, 0, 0)$$

for all $x \in X$.

Hence

$$\begin{aligned} \left\|\frac{1}{a^{2l}}f(a^l x) - \frac{1}{a^{2m}}f(a^m x)\right\| &\leq \sum_{j=l+1}^{m}\left\|\frac{1}{a^{2(j-1)}}f(a^{j-1}x) - \frac{1}{a^{2j}}f(a^j x)\right\| \\ &\leq \sum_{j=l+1}^{m}\frac{1}{4a^{2j}}\phi(a^{j-1}x, 0, 0) \qquad (2.13) \end{aligned}$$

for all $x \in X$. It means that a sequence $\{\frac{1}{a^{2n}}f(a^n x)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{a^{2n}}f(a^n x)\}$ converges. So one can define a mapping $Q : X \to Y$ by $Q(x) := \lim_{n\to\infty}\frac{1}{a^{2n}}f(a^n x)$ for all $x \in X$.

By (2.10) and (2.11),

$$\begin{aligned} \|DQ(x, y, z)\| &= \lim_{n\to\infty}\frac{1}{a^{2n}}\|Df(a^n x, a^n y, a^n z)\| \\ &\leq \lim_{n\to\infty}\frac{1}{a^{2n}}\phi(a^n x, a^n y, a^n z) = 0 \end{aligned}$$

for all $x, y, z \in X$. So $DQ(x, y, z) = 0$. By Lemma 2.1, the mapping $Q : X \to Y$ is a quadratic.

Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.13), we get the approximation (2.12) of f by Q.

Now, let $Q' : X \longrightarrow Y$ be another quadratic mapping satisfying (2.12) . Then we obtain

$$\begin{aligned}\|Q(x)-Q'(x)\| &= \frac{1}{a^{2n}}\|Q(a^n x)-Q'(a^n x)\| \\ &\leq \frac{1}{a^{2n}}[\|Q(a^n x)-f(a^n x)\|+\|Q'(a^n x)-f(a^n x)\|] \\ &\leq \frac{1}{2a^{2(n+1)}}\Phi(a^n x,0,0),\end{aligned}$$

which tends to zero as $n \to \infty$. So we can conclude that $Q(x) = Q'(x)$ for all $x \in X$. This proves the uniqueness of Q. Hence the mapping $Q : X \to Y$ is a unique quadratic mapping satisfying (2.12). □

Corollary 2.5. *Let p and θ be positive real numbers with either $p < 2$ and $|a| > 1$ or $p > 2$ and $|a| < 1$, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\|Df(x,y,z)\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p), \tag{2.14}$$

for all $x, y, z \in X$. Then there exists a unique Euler-Lagrange type quadratic mapping $Q : X \to Y$ such that

$$\|f(x)-Q(x)\| \leq \frac{\theta\cdot\|x\|^p}{4(a^2-|a|^p)} \tag{2.15}$$

for all $x \in X$.

Proof. Define $\phi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$, and apply Theorem 2.4. □

Corollary 2.6. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a nonnegative number θ such that*

$$\|Df(x,y,z)\| \leq \theta \tag{2.16}$$

for all $x, y, z \in X$. If $|a| \neq 1$, then there exists a unique Euler-Lagrange type quadratic mapping $Q : X \to Y$ such that

$$\|f(x)-Q(x)\| \leq \frac{\theta}{4|1-a^2|} \tag{2.17}$$

for all $x \in X$.

Proof. Define $\phi(x, y, z) = \theta$, and apply Theorem 2.2 and Theorem 2.4 . □

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