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STABILITY OF A GENERALIZED JENSEN TYPE QUADRATIC FUNCTIONAL EQUATIONS

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Abstract. In this paper, we investigate the Hyers–Ulam–Rassias stability of generalized Jensen type quadratic functional equations in Banach spaces.

1. Introduction

In 1940, S. M. Ulam [1] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

Let G be a group and let G' be a metric group with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

In 1941, D. H. Hyers [2] considered the case of an approximately additive mapping $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon$$

for all $x, y \in E$. It was shown that the limit $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \le \epsilon.$$

Let E_1 and E_2 be real vector spaces. A function $f : E_1 \to E_2$, there exists a quadratic function if and only if f is a solution function of the

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quadratic functional equation

$$f(x+y) - f(x-y) = 2f(x) + 2f(y). \tag{1.1}$$

A stability problem for the quadratic functional equation (1.1) was solved by F. Skof [3] for a mapping $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space.

In 1978, Th. M. Rassias [4] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

S. Czerwik [5] proved the Hyers-Ulam-Rassias stability of quadratic functional equation (1.1). Let E_1 and E_2 be a real normed space and a real Banach space, respectively, and let $p \neq 2$ be a positive constant. If a function $f : E_1 \to E_2$ satisfies the inequality

$$\|f(x+y) - f(x-y) - 2f(x) - 2f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for some $\epsilon > 0$ and for all $x, y \in E_1$, then there exists a unique quadratic function $q : E_1 \to E_2$ such that

$$\|f(x) - q(x)\| \leq \frac{2\epsilon}{|4 - 2^p|}\|x\|^p$$

for all $x \in G$.

A mapping $g : X \to Y$ is called a *Jensen mapping* if g satisfies the functional equation

$$2g(\frac{x+y}{2}) = g(x) + g(y)$$

for all $x, y \in X$.

Jun and Lee [6] proved the following : Let X and Y be Banach spaces. Denote by $\varphi : X \setminus \{0\} \times X \setminus \{0\} \to [0, \infty)$ function such that

$$\psi(x, y) = \sum_{j=0}^{\infty} 3^{-k}\varphi(3^k x, 3^k y) < \infty$$

for all $x, y \in X \setminus \{0\}$. Suppose that $f : X \to Y$ is a mapping satisfying

$$\|2f(\frac{x+y}{2}) - f(x) - f(y)\| \leq \varphi(x, y)$$

for all $x, y \in X \setminus \{0\}$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - f(0) - T(x)\| \leq \frac{1}{3}(\psi(x, -x) + \psi(-x, 3x))$$

for all $x \in X \setminus \{0\}$. Recently C. Park and W. Park [7] applied the Jun and Lee's result to the Jensen's equation in Banach modules over a C^*-algebra.

Now, we consider the following functional equation

$$f(\frac{x+y}{a}+bz) + f(\frac{x+y}{a}-bz)+f(\frac{x-y}{a}+bz)+f(\frac{x-y}{a}-bz) = \frac{4}{a^2}f(x)+\frac{4}{a^2}f(y)+4b^2f(z), \tag{1.2}$$

where $a, b \neq 0$ are real numbers.

In this paper, we will establish the general solution and the generalized Hyers-Ulam-Rassias stability problem for the equation (1.2) in Banach spaces.

2. Jesen type quadratic mapping in Banach spaces

LEMMA 2.1. *Let X and Y be vector spaces. If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$f(\frac{x+y}{a}+bz)+f(\frac{x+y}{a}-bz)+f(\frac{x-y}{a}+bz)+f(\frac{x-y}{a}-bz) = \frac{4}{a^2}f(x)+\frac{4}{a^2}f(y)+4b^2f(z) \tag{2.1}$$

for all $x, y, z \in X$, then the mapping f is quadratic.

Proof. Letting $x = y$ in (2.1), we get

$$f(\frac{2x}{a}+bz)+f(\frac{2x}{a}-bz)+f(bz)+f(-bz) = \frac{8}{a^2}f(x)+4b^2f(z) \tag{2.2}$$

for all $x, z \in X$. Letting $x = 0$ in (2.2), we get $2f(bz)+2f(-bz) = 4b^2f(z)$. Setting $y = -x$ in (2.1), we obtain

$$f(\frac{2x}{a}+bz)+f(\frac{2x}{a}-bz)+f(bz)+f(-bz) = \frac{4}{a^2}f(x)+\frac{4}{a^2}f(-x)+4b^2f(z). \tag{2.3}$$

By (2.2) and (2.3), we conclude that f is even. And by setting $z = 0$ in (2.2), we get $f(\frac{2x}{a}) = \frac{4}{a^2}f(x)$ for all $x \in X$. So, we get

$$f(\frac{2x}{a}+bz)+f(\frac{2x}{a}-bz) = 2f(\frac{2x}{a})+2f(bz)$$

for all $x, z \in X$. Hence f is quadratic. □

The mapping $f : X \to Y$ given in the statement of Lemma 2.1 is called a generalized *Jensen type quadratic mapping*. Putting $z = 0$ in (2.1) with $a = 2$, we get the Jensen type quadratic mapping $2f(\frac{x+y}{2}) + 2f(\frac{x-y}{2}) = f(x)+f(y)$, and putting $x = y$ in (2.1) with $a = 2$ and $b = 1$, we get the quadratic mapping $f(x+z)+f(x-z) = 2f(x)+2f(z)$.

From now on, let X and Y be a normed vector space and a Banach space ,respectively.

For a given mapping $f : X \to Y$, we define

$$\begin{aligned} Df(x,y,z) := f(\frac{x+y}{a} + bz) + f(\frac{x+y}{a} - bz) + f(\frac{x-y}{a} + bz) \\ + f(\frac{x-y}{a} - bz) - \frac{4}{a^2}f(x) - \frac{4}{a^2}f(y) - 4b^2 f(z) \end{aligned}$$

for all $x, y, z \in X$

THEOREM 2.2. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^3 \to [0,\infty)$ such that*

$$\Phi(x,y,z) := \sum_{j=1}^{\infty} \frac{1}{2}\big(\frac{4}{a^2}\big)^j \phi\big(\big(\frac{a}{2}\big)^j x, \big(\frac{a}{2}\big)^j y, \big(\frac{a}{2}\big)^j z\big) < \infty, \tag{2.4}$$

$$\|Df(x,y,z)\| \le \phi(x,y,z) \tag{2.5}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that $DQ(x,y,z) = 0$ and

$$\|Q(x) - f(x)\| \le \frac{a^2}{4}\Phi(x,x,0) \tag{2.6}$$

for all $x \in X$.

Proof. Letting $x = y$ and $z = 0$ in (2.5), we get

$$\|f\big(\frac{2}{a}x\big) - \frac{4}{a^2}f(x)\| \le \frac{1}{2}\phi(x,x,0)$$

for all $x \in X$. So

$$\|f(x) - \frac{4}{a^2}f\big(\frac{a}{2}x\big)\| \le \frac{1}{2}\phi\big(\frac{a}{2}x, \frac{a}{2}x, 0\big)$$

for all $x \in X$. Hence

$$\|(\frac{4}{a^2})^l f((\frac{a}{2})^l x) - (\frac{4}{a^2})^m f((\frac{a}{2})^m x)\|$$
$$\leq \sum_{j=l+1}^{m} \|(\frac{4}{a^2})^{j-1} f((\frac{a}{2})^{j-1} x) - (\frac{4}{a^2})^j f((\frac{a}{2})^j x)\|$$
$$\leq \sum_{j=l+1}^{m} (\frac{4}{a^2})^{j-1} \frac{1}{2}\phi((\frac{a}{2})^j x, (\frac{a}{2})^j x, 0)$$

for all $x \in X$. It means that a sequence $\{(\frac{4}{a^2})^n f((\frac{a}{2})^n x)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{(\frac{4}{a^2})^n f((\frac{a}{2})^n x)\}$ converges. So one can define a mapping $Q : X \to Y$ by $Q(x) := \lim_{n\to\infty}(\frac{4}{a^2})^n f((\frac{a}{2})^n x)$ for all $x \in X$.

By (2.4) and (2.5),

$$\|DQ(x,y,z)\| = \lim_{n\to\infty} (\frac{4}{a^2})^n \|Df((\frac{a}{2})^n x, (\frac{a}{2})^n y, (\frac{a}{2})^n z)\| \tag{2.7}$$
$$\leq \lim_{n\to\infty} (\frac{4}{a^2})^n \phi((\frac{a}{2})^n x, (\frac{a}{2})^n y, (\frac{a}{2})^n z) = 0 \tag{2.8}$$

for all $x, y, z \in X$. So $DQ(x,y,z) = 0$. By Lemma 2.1, the mapping $Q : X \to Y$ is a quadratic mapping.

Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.7), we get the approximation (2.6) of f by Q.

Now, let $Q' : X \longrightarrow Y$ be another quadratic mapping satisfying (2.6). Then we obtain

$$\|Q(x) - Q'(x)\| = (\frac{4}{a^2})^n \|Q((\frac{a}{2})^n x) - Q((\frac{a}{2})^n x)\|$$
$$\leq (\frac{4}{a^2})^n [\|Q((\frac{a}{2})^n x) - f((\frac{a}{2})^n x)\| + \|Q'((\frac{a}{2})^n x) - f((\frac{a}{2})^n x))\|]$$
$$\leq 2(\frac{4}{a^2})^{n-1} \Phi((\frac{a}{2})^n x, (\frac{a}{2})^n x, 0),$$

which tends to zero as $n \to \infty$. So we can conclude that $Q(x) = Q'(x)$ for all $x \in X$. This proves the uniqueness of Q. Hence the mapping $Q : X \to Y$ is a unique quadratic mapping satisfying (2.6). □

COROLLARY 2.3. *Let p and θ be positive real numbers with $p > 2$ and $0 < |a| < 2$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\|Df(x,y,z)\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p) \tag{2.9}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{|a|^p \cdot \theta}{4(2^{p-2} - |a|^{p-2})}\|x\|^p \tag{2.10}$$

for all $x \in X$.

Proof. Define $\phi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$, and apply Theorem 2.2. □

COROLLARY 2.4. *Let p and θ be positive real numbers with $p < 2$ and $|a| > 2$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\|Df(x, y, z)\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p) \tag{2.11}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{|a|^2 \cdot \theta}{2^p(|a|^{2-p} - 2^{2-p})}\|x\|^p \tag{2.12}$$

for all $x \in X$.

Proof. Define $\phi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$, and apply Theorem 2.2. □

THEOREM 2.5. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^3 \to [0, \infty)$ such that*

$$\begin{aligned}\Phi(x, y, z) := &\sum_{j=1}^{\infty} \frac{1}{2}\Big(\frac{a^2}{4}\Big)^j \phi\Big(\Big(\frac{2}{a}\Big)^{j-1}x, \Big(\frac{2}{a}\Big)^{j-1}y, \Big(\frac{2}{a}\Big)^{j-1}z\Big) \\ &< \infty,\end{aligned} \tag{2.13}$$

$$\|Df(x, y, z)\| \leq \phi(x, y, z) \tag{2.14}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that $DQ(x, y, z) = 0$ and

$$\|Q(x) - f(x)\| \leq \Phi(x, x, 0) \tag{2.15}$$

for all $x \in X$.

Proof. Letting $x = y$ and $z = 0$ in (2.5), we get

$$\|f\Big(\frac{2}{a}x\Big) - \frac{4}{a^2}f(x)\| \leq \frac{1}{2}\phi(x, x, 0) \tag{2.16}$$

for all $x \in X$. So

$$\|f(x) - \frac{a^2}{4}f\Big(\frac{2}{a}x\Big)\| \leq \Big(\frac{a^2}{4}\Big)\frac{1}{2}\phi(x, x, 0) \tag{2.17}$$

for all $x \in X$.

Hence

$$\|(\frac{a^2}{4})^l f((\frac{2}{a})^l x) - (\frac{a^2}{4})^m f((\frac{2}{a})^m x)\| \leq \sum_{j=l+1}^{m} (\frac{a^2}{4})^j \frac{1}{2}\phi((\frac{2}{a})^{j-1}x, (\frac{2}{a})^{j-1}x, 0) \tag{2.18}$$

for all $x \in X$. It means that a sequence $\{(\frac{a^2}{4})^n f((\frac{2}{a})^n x)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{(\frac{a^2}{4})^n f((\frac{2}{a})^n x)\}$ converges. So one can define a mapping $Q : X \to Y$ by $Q(x) := \lim_{n\to\infty}(\frac{a^2}{4})^n f((\frac{2}{a})^n x)$ for all $x \in X$.

By (2.13) and (2.14),

$$\|DQ(x,y,z)\| = \lim_{n\to\infty} (\frac{a^2}{4})^n \|Df((\frac{2}{a})^n x, (\frac{2}{a})^n y, (\frac{2}{a})^n z)\| \tag{2.19}$$

$$\leq \lim_{n\to\infty} (\frac{a^2}{4})^n \phi((\frac{2}{a})^n x, (\frac{2}{a})^n y, (\frac{2}{a})^n z) = 0 \tag{2.20}$$

for all $x, y, z \in X$. So $DQ(x,y,z) = 0$. By Lemma 2.1, the mapping $Q : X \to Y$ is a quadratic mapping.

Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.18), we get the approximation (2.15) of f by Q.

Now, let $Q' : X \longrightarrow Y$ be another quadratic mapping satisfying (2.15) . Then we obtain

$$\begin{aligned}
&\|Q(x) - Q'(x)\| = (\frac{a^2}{4})^n \|Q((\frac{2}{a})^n x) - Q((\frac{2}{a})^n x)\| \\
&\leq (\frac{a^2}{4})^n [\|Q((\frac{2}{a})^n x) - f((\frac{2}{a})^n x)\| + \|Q'((\frac{2}{a})^n x) - f((\frac{2}{a})^n x))\|] \\
&\leq 2(\frac{a^2}{4})^n \Phi((\frac{2}{a})^n x, (\frac{2}{a})^n x, 0),
\end{aligned}$$

which tends to zero as $n \to \infty$. So we can conclude that $Q(x) = Q'(x)$ for all $x \in X$. This proves the uniqueness of Q. Hence the mapping $Q : X \to Y$ is a unique quadratic mapping satisfying (2.15). □

Corollary 2.6. *Let p and θ be positive real numbers with $p > 2$ and $|a| > 2$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\|Df(x,y,z)\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p) \tag{2.21}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{|a|^p \cdot \theta}{4(|a|^{p-2} - 2^{p-2})}\|x\|^p \tag{2.22}$$

for all $x \in X$.

Proof. Define $\phi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$, and apply Theorem 2.5. □

COROLLARY 2.7. *Let p and θ be positive real numbers with $p < 2$ and $|a| < 2$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\|Df(x, y, z)\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p) \tag{2.23}$$

for all $x, y, z \in X$. Then there exists a unique Jensen type quadratic-quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{|a|^2 \cdot \theta}{2^p(2^{2-p} - |a|^{2-p})}\|x\|^p \tag{2.24}$$

for all $x \in X$.

Proof. Define $\phi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$, and apply Theorem 2.5. □

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