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THE RELATIVE REGIONALLY REGULAR RELATION

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Abstract. In this paper the relative regionally regular relation is defined and it will be given the necessary and sufficient conditions for the relation to be an equivalence relation.

1. Introduction

The proximal relation P and the regionally proximal relation Q of a transformation group (X, T) were first defined by Ellis and Gottschalk ([6]), and syndetically proximal relation L and the syndetically regionally proximal relation M were introduced and studied intensively by Clay([3]). The author ([9]) introduced the regular relation, regionally regular relation and syndetically regionally regular relation as the generalizations of those of Ellis, Gottschalk and Clay.

A homomorphism $\pi : X \to Y$ determines a closed invariant equivalence relation $R_\pi = \{(x, x') \in X \times X \mid \pi(x) = \pi(x')\}$ on X. Conversely, a closed invariant equivalence relation R on X determines an epimorphism $\pi : X \to X/R$. Given $\pi : X \to Y$, Shoenfeld ([8]) introduced the relative proximal relation on X to be the intersection of R_π with the proximal relation on X. He also defined the relative regionally proximal relation on X.

In this paper, we will define the relative regionally regular relation and the relative syndetically regionally regular relation.

2. Preliminaries

Throughout this paper, (X, T) will denote the transformation group with compact Hausdorff phase space X. A closed nonempty subset A of

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X is called a *minimal subset* if for every $x \in A$, the orbit xT is a dense subset of A. A point whose orbit closure is a minimal set is called an *almost periodic point.* If X is itself minimal, we say that it is a *minimal transformation group* or a *minimal set.*

The compact Hausdorff space X carries a natural uniformity whose indicies are the neighborhoods of the diagonal in $X \times X$. Two points x and y are called *proximal* provided that for each index α of X, there exists a $t \in T$ such that $(xt, yt) \in \alpha$. The set of all proximal pairs is called the *proximal relation* and is denoted by $P(X)$, or simply P. If x and y are not proximal, x and y are said to be *distal.* If $P(X) = \Delta$, the digonal of $X \times X$, then (X,T) is called the *distal transformation group.*

A continuous map $\pi : (X,T) \to (Y,T)$ with $\pi(xt) = \pi(x)t$ is called a *homomorphism.* If Y is minimal, π is always onto. If π is onto, π is called an *epimorphism* A homomorphism of X into itself is called an *endomorphism.* An isomorphism $\pi : X \to Y$ is called an *automorphism* of X. We denote the group of automorphisms of X by $A(X)$.

A minimal transformation group (X,T) is called *regular* if for every almost periodic point $(x,y) \in (X \times X, T)$, there exists an endomorphism h of (X,T) such that $h(x) = y$

Definition 2.1. ([8],[9]) (1) Two points x and y of (X,T) are said to be *regionally proximal* if there exist nets (x_n) and (y_n) in X, a net (t_n) in T and a point $x'' \in X$ such that $\lim x_n = x$, $\lim y_n = y$ and $\lim x_n t_n = \lim y_n t_n = x''$. The set of regionally proximal pairs of points is called the *regionally proximal relation* and is denoted by $Q(X)$ or Q.

(2) Let Q^* denote the set of all $(x,y) \in X \times X$ such that $(h(x), y) \in Q$ for some $h \in A(X)$. The set Q^* is called the *regionally regular relation.*

Definition 2.2. ([2],[9]) (1) Let $M(X)$ or simply M denote the set of all pairs $(x,y) \in X \times X$ such that for every index α and for every neighborhood U of x, every neighborhood V of y, there exist points $x_1 \in U$ and $y_1 \in V$ and a syndetic subset A of T such that $(x_1 t, y_1 t) \in \alpha$ for all $t \in A$. The set $M(X)$ is called the *syndetically regionally proximal relation.*

(2) Let M^* denote the set of all $(x,y) \in X \times X$ such that $(h(x), y) \in M$ for some $h \in A(X)$. The set M^* is called *syndetically regionally regular relation.*

Let $\pi : X \to Y$ be a homomorphism and let

$$R_\pi = \{(x,x') \in X \times X \mid \pi(x) = \pi(x')\}$$

Then R_π is a closed invariant equivalence relation on X.

Definition 2.3. ([8],[2])(1) Let $Q(\pi)$ denote the set of all pairs $(x, y) \in R_\pi$ such that for every index α there are points $x_1 \in x\alpha$, $y_1 \in y\alpha$ and an element $t \in T$ satisfying the condition $(x_1, y_1) \in R_\pi$ and $(x_1 t, y_1 t) \in \alpha$. Equivalently, $Q(\pi)$ is the set of all $(x, x') \in R_\pi$ such that there exist nets (x_n) and (x'_n) in X, a net (t_n) in T, and a point $x'' \in X$ satisfying the condition $(x_n, x'_n) \in R_\pi$ and $\lim x_n = x$, $\lim x'_n = x'$ and $\lim x_n t_n = x''$, $\lim x'_n t_n = x''$. The set $Q(\pi)$ is called *the relative regionally proximal relation.*

(2) Let $M(\pi)$ denote the set of all pairs $(x, y) \in R_\pi$ such that for every index α there exist points $x_1 \in x\alpha$, $y_1 \in y\alpha$ and a syndetic subset $A \subset T$ such that $(x_1, y_1) \in R_\pi$ and $(x_1 t, y_1 t) \in \alpha$ for all $t \in A$. The set $M(\pi)$ is called the *relative syndetically regionally proximal relation.*

Remark 2.4. (1) If $R_\pi = X \times X$, then $Q(\pi)$ coincides with the regionally proximal relation Q and $M(\pi)$ coincides with the syndetically regionally proximal relation M.

(2) It is obvious that $Q(\pi)$ and $M(\pi)$ are invariant closed relations and $M(\pi) \subset Q(\pi) \subset R_\pi$.

Theorem 2.5. *([2]) If (X, T) is a distal transformation group, then $M(\pi) = Q(\pi)$.*

Theorem 2.6. *([2])If (X, T) is a distal minimal transformation group, then $Q(\pi)$ is a closed invariant equivalence relation.*

3. Relative regionally regular relation

Now, we introduce the relative regionally regular relation and relative syndetically regionally regular relation as the generalized notions of the regionally regular relation and the syndetically regionally regular relation.

Definition 3.1. Let $\pi : X \to Y$ be a homomorphism and let $R_\pi = \{(x, x') \mid \pi(x) = \pi(x')\}$. The sets $Q^*(\pi)$ and $M^*(\pi)$ are defined as follows.

(1) $(x, y) \in Q^*(\pi)$ if and only if $(h(x), y) \in Q(\pi)$ for some $h \in A(X)$.

(2) $(x, y) \in M^*(\pi)$ if and only if $(h(x), y) \in M(\pi)$ for some $h \in A(X)$.

The sets $Q^*(\pi)$ and $M^*(\pi)$ are called *the relative regionally regular relation* and *the relative syndetically regionally regular relation* on X, respectively.

Remark 3.2. (1) If $R_\pi = X \times X$, then $Q^*(\pi) = Q^*$ and $M^*(\pi) = M^*$

(2) $Q^*(\pi)$ and $M^*(\pi)$ are invariant reflexive relations.

THEOREM 3.3. *(1) If $M(\pi) = Q(\pi)$, then $M^*(\pi) = Q^*(\pi)$.*
(2) If (X, T) is a distal transformation group, then $M^*(\pi) = Q^*(\pi)$.

Proof. (1) Suppose that $M(\pi) = Q(\pi)$. It suffices to show that $M^*(\pi) \subset Q^*(\pi)$. Let $(x, y) \in Q^*(\pi)$. Then $(h(x), y) \in Q(\pi)$ for some $h \in A(X)$. Since $M(\pi) = Q(\pi)$, $(h(x), y) \in M(\pi)$. This shows that $(x, y) \in M^*(\pi)$.

(2) In a distal transformation group (X, T), $M(\pi) = Q(\pi)$ by Theorem 2.5. Therefore, $M^*(\pi) = Q^*(\pi)$ by (1). □

DEFINITION 3.4. An epimorphism $\pi : X \to Y$ is called an *r-homomorphism* if given $h \in A(X)$, there exists a $k \in A(Y)$ such that $\pi h = k\pi$.

THEOREM 3.5. *Let (X, T) be minimal and let (Y, T) be regular minimal. Then $\pi : X \to Y$ is an r-homomorphism.*

Proof. Let $h \in A(X)$ and $y \in Y$. Since Y is minimal, there exists $v^2 = v$ in $E(Y)$ such that $yv = y$. There exists an almost periodic point $x = xu$ of X such that $u^2 = u$, $\tilde{\pi}(u) = v$ and $\pi(x) = \pi(xu) = \pi(x)\tilde{\pi}(u) = \pi(x)v = yv = y$, where $\tilde{\pi} : E(X) \to E(Y)$ is the homomorphism induced by $\pi : X \to Y$.

Since $(y, \pi h(x)) = (yv, \pi h(xu)) = (yv, \pi h(x)v) = (y, \pi h(x))v$, $(y, \pi h(x))$ is an almost periodic point of $Y \times Y$. Since Y is regular minimal, there exists $k \in A(Y)$ such that $k(y) = \pi h(x)$. that is, $k\pi(x) = \pi h(x)$. Since a homomorphism of minimal sets is determined by its value at one point, it follows that $k\pi = \pi h$, and therefore π is an r-homomorphism. □

THEOREM 3.6. *Let (X, T) and (Y, T) be minimal and let $\pi : X \to Y$ be an r-homomorphism. If X is regular, then so is Y.*

Proof. Let X be regular minimal and let $\pi : X \to Y$ be an r-homomorphism. We will show that for any almost periodic point (y, y') of $(Y \times Y, T)$, there is an endomorphism k of (Y, T) such that $k(y) = y'$. Let (y, y') be an almost periodic point of $(Y \times Y, T)$. Then there is an almost periodic point (x, x') in $(X \times X, T)$ such that

$$\pi^*(x, x') = (\pi(x), \pi(x')) = (y, y'),$$

where homomorphism $\pi^* : X \times X \to Y \times Y$ is defined by $\pi^*(x, x') = (\pi(x), \pi(x'))$. Since X is regular minimal and (x, x') is an almost periodic point of $Y \times Y$, there exists an endomorphism h of X such that $h(x) = x'$. But, endomorphism h is, in fact, an automorphism, because every endomorphism in a regular minimal set is an automorphism. Since π is an r-homomorphism, given $h \in A(X)$, there exists a $k \in A(Y)$ such

that $\pi h = k\pi$.
Hence,

$$k(y) = k\pi(x) = \pi h(x) = \pi(x') = y'.$$

Therefore, Y is regular minimal. $\square$

REMARK 3.7. *Let $\pi : X \to Y$ be an r-homomorphism, and let $(x, x') \in Q(\pi)$. Then $(h(x), h(x')) \in Q(\pi)$ for an automorphism h of X.*

Proof. Let $(x, x') \in Q(\pi)$ and let h be an automorphism of X. There are nets (x_n),(x'_n) in X, a net (t_n) in T, and a point $x'' \in X$ such that each $(x_n, x'_n) \in R_\pi$ and

$$\lim x_n = x,\ \lim x'_n = x' \text{ and } \lim x_n t_n = x'',\ \lim x'_n t_n = x''.$$

Put $y_n = h(x_n)$, $y'_n = h(x'_n)$ for each n. Then (y_n), (y'_n) are nets in X and it follow that

$$\lim y_n = \lim h(x_n) = h(x),\ \lim y'_n = \lim h(x'_n) = h(x')$$

and

$$\lim y_n t_n = \lim h(x_n)t_n = \lim h(x_n t_n) = h(x'')$$
$$\lim y'_n t_n = \lim h(x'_n)t_n = \lim h(x'_n t_n) = h(x'')$$

Since π is an r-homomorphism, given $h \in A(X)$ there exists a $k \in A(Y)$ such that $\pi h = k\pi$. Hence we obtain

$$\pi(y_n) = \pi h(x_n) = k\pi(x_n) = k\pi(x'_n) = \pi h(x'_n) = \pi(y'_n).$$

that is, each $(y_n, y'_n) \in R_\pi$ and therefore, $(h(x), h(x')) \in Q(\pi)$. $\square$

THEOREM 3.8. *Let $\pi : X \to Y$ be an r-homomorphism. If the relative regionally proximal relation $Q(\pi)$ is an equivalence relation, then so is the relative regionally regular relation $Q^*(\pi)$.*

Proof. Suppose that $Q(\pi)$ is an equivalence relation. It is obvious that $Q^*(\pi)$ is reflexive. Let $(x, y) \in Q^*(\pi)$. Then $(h(x), y) \in Q(\pi)$ for some $h \in A(X)$. Since $Q(\pi)$ is symmetric, $(y, h(x)) \in Q(\pi)$. We also have $(h^{-1}(y),\ h^{-1}h(x)) = (h^{-1}(y), x) \in Q(\pi)$ by Lemma 3.7. This implies that $(y, x) \in Q^*(\pi)$. Therefore, $Q^*(\pi)$ is symmetric. Next we show that $Q^*(\pi)$ is transitive. $(x, y) \in Q^*(\pi)$ and $(y, z) \in Q^*(\pi)$. Then $(h(x), y) \in Q(\pi)$ and $(k(y), z) \in Q(\pi)$ for some h, k in $A(X)$. We also have $(kh(x), k(y)) \in Q(\pi)$ by Lemma 3.7. Since $Q(\pi)$ is transitive, $(kh(x), z) \in Q(\pi)$. This shows that $(x, z) \in Q^*(\pi)$. Therefore, $Q^*(\pi)$ is transitive. $\square$

By Theorem 2.6, the relative rigionally proximal relation $Q(\pi)$ in a distal minimal transformation group (X, T) is an invariant equivalence relation. Consequently, we have the following corollary.

COROLLARY 3.9. *Let $\pi : X \to Y$ be an r-homomorphism and let (X, T) be a distal minimal transformation group. Then $Q^*(\pi)$ is an equivalence relation.*

For a fixed $h \in A(X)$, we denote $Q^*_h(\pi)$ to be the set of all $(x, y) \in R_\pi$ satisfying the condition $(h(x), y) \in Q(\pi)$.

It is obvious that $Q^*(\pi) = \cup\{Q^*_h(\pi) \mid h \in A(X)\}$ and $Q(\pi) = Q^*_1(\pi)$, where 1 is the identity automorphism of X.

THEOREM 3.10. *Let $\pi : X \to Y$ be a homomorphism. Then $Q^*_h(\pi)$ is a closed subset of $X \times X$ for all $h \in A(X)$.*

Proof. Let $((x_n, y_n))$ be a set in $Q^*_h(\pi)$ such that $((x_n, y_n))$ converges to (x, y). Then $(h(x_n), y_n) \in Q(\pi)$ for all n, and $(h(x_n), y_n)$ converges to $(h(x), y)$. Since $Q(\pi)$ is closed, $(h(x), y) \in Q(\pi)$, and therefore $(x, y) \in Q^*_h(\pi)$. This shows that $Q^*_h(\pi)$ is closed in $X \times X$. □

The following corollary is immedinte from the fact that $Q(\pi) = Q^*_1(\pi)$.

COROLLARY 3.11. *Let $\pi : X \to Y$ be a homomorphism. Then the relative regionally proximal relation $Q(\pi)$ is closed in $X \times X$.*

The following theorem is an analogue of Theorem 3.15([9]).

THEOREM 3.12. *Let $\pi : X \to Y$ be an r-homomorphism. Then*

(1) $Q(\pi)$ is an equivalence relation if and only if $Q^*_h(\pi) \circ Q^*_k(\pi) = Q^*_{hk}(\pi)$ for all h, k in $A(X)$.

(2) $Q^*(\pi)$ is an equivalence relation if and only if $Q^*_h(\pi) \circ Q^*_k(\pi) \subset Q^*(\pi)$ for all h, k in $A(X)$.

Proof. The proof is similar to that of Theorem 3.15([9]), and therefore will be omitted. □

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