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ALMOST PERIODIC SOLUTIONS OF FUNCTIONAL DIFFERENCE EQUATIONS WITH FINITE DELAY

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Abstract. We obtain a discrete version of a fundamental result by Yoshizawa related to the existence of almost periodic solutions of functional differential equations with finite delay.

1. Introduction

The term functional differential equation is a natural generalization of the concept of a differential equation with delayed arguments. Its evolution involves past values of the state variable. Thus its solution requires knowledge of not only the current state, but also of the state certain time previously. Time delays are commonly encountered in many physical systems and control systems due to the finite switching times, network traffic congestions, etc.

Functional difference equation is a discrete analogue of functional differential equation. The following functional difference equation with finite delay :

$$x(n+1) = f(n, x_n),\ n \geq 0, \tag{1.1}$$

under certain conditions for $f(n,\cdot)$, and x_n denotes the function $x(n+\tau), \tau = -r, -r+1, \cdots, 0,\ \ n, \tau, r$ are integers, can be regarded as the discrete analogue of the functional differential equation with bounded delay :

$$x'(t) = F(t, x_t),\ x_t(0) = x(t+0) = \phi(t),\ -r \leq t \leq 0. \tag{1.2}$$

It is interesting to study how some results for functional differential equations can be discretized to cover such equations. The purpose of this

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paper is to obtain a discrete analogue of a result in [3] related to the existence of almost periodic solutions of functional differential equation (1.2).

2. Preliminaries

We denote by $\mathbb{Z}$ and $\mathbb{Z}_+$, respectively, the set of integers, and the set of nonnegative integers. $\mathbb{R}^n$ is the n-dimensional real Euclidean space with the suitable norm $|\cdot|$.

We recall the definitions and the basic properties of almost periodic sequences in [4] and [5].

Definition 2.1. *A function $f : \mathbb{R} \to \mathbb{R}^n$ is almost periodic if the ε-translation set of f*

$$E\{\varepsilon, f\} = \{\tau \in \mathbb{R} : |f(t+\tau) - f(t)| < \varepsilon, t \in \mathbb{R}\}$$

is relatively dense in $\mathbb{R}$ for all $\varepsilon > 0$, i.e., for any $\varepsilon > 0$ there exists a number $l = l(\varepsilon) > 0$ with the property that every interval of length l contains a $\tau \in \mathbb{R}$ such that

$$|f(t+\tau) - f(t)| < \varepsilon, t \in \mathbb{R}.$$

Definition 2.2. *A sequence $x : \mathbb{Z}_+ \to \mathbb{R}^n$ is almost periodic if for any $\varepsilon > 0$ there exists a number $l = l(\varepsilon) > 0$ for which every discrete interval of length l contains a $\tau \in \mathbb{Z}$ such that*

$$|f(n+\tau) - f(n)| < \varepsilon, n \in \mathbb{Z}.$$

Definition 2.3. *Let $f : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous in $x \in \mathbb{R}^n$ for each $n \in \mathbb{Z}$. Then f is almost periodic in $n \in \mathbb{Z}$ uniformly in $x \in \mathbb{R}^n$ or uniformly almost periodic if for every $\varepsilon > 0$ and every compact set K in $\mathbb{R}^n$ corresponds an integer $l = l(\varepsilon, K) > 0$ such that among l consecutive integers there is one τ such that*

$$|f(n+\tau, x) - f(n, x)| < \varepsilon, n \in \mathbb{Z}, x \in K.$$

For uniform almost periodic sequences, we have the following results in [2].

Theorem 2.4. *Let $f : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^n$ be a uniformly almost periodic function. Then for any integer sequence (τ'_k) with $\tau'_k \to \infty$ as $k \to \infty$, there exist a subsequence $(\tau_k) \subset (\tau'_k), \tau_k \to \infty$ as $k \to \infty$, and a function $g : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$f(n + \tau_k, x) \to g(n, x)$$

uniformly on $\mathbb{Z} \times K$ *as* $k \to \infty$*, where* K *is any compact set in* $\mathbb{R}^n$*. Moreover,* $g(n,x)$ *is also uniformly almost periodic.*

Theorem 2.5. *If* $f : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^n$ *is a uniformly almost periodic function, then there exists a sequence* $(\tau_k) \subset \mathbb{Z}, \tau_k \to \infty$ *as* $k \to \infty$*, such that*

$$f(n+\tau_k, x) \to f(n,x),$$

uniformly on $\mathbb{Z} \times K$ *as* $k \to \infty$*, where* K *is any compact set in* $\mathbb{R}^n$*.*

Theorem 2.6. [4] *Let* $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *be continuous in* $x \in \mathbb{R}^n$ *for each* $t \in \mathbb{R}^n$ *and uniformly almost periodic. If* $x : \mathbb{R} \to S$ *is an almost periodic function, where* S *is a compact subset in* $\mathbb{R}^n$*, then* $f(t, x(t))$ *is uniformly almost periodic.*

As a generalization of almost periodicity, we give the following definition [5].

Definition 2.7. *A sequence* $x : \mathbb{Z}_+ \to \mathbb{R}^n$ *is called asymptotically almost periodic if* $x(n) = p(n)+q(n)$ *for all* $n \in \mathbb{Z}_+$*, where* $p(n)$ *is almost periodic and* $q(n) \to 0$ *as* $n \to \infty$*.*

Theorem 2.8. [2] *Let* $x : \mathbb{Z}_+ \to \mathbb{R}^n$*. Then the following statements are equivalent.*

(i) *x is asymptotically almost periodic.*
(ii) *For any sequence* $(\tau'_k) \subset \mathbb{Z}_+, \tau'_k > 0$ *and* $\tau'_k \to \infty$ *as* $k \to \infty$*, there exists a subsequence* (τ_k) *of* (τ'_k) *such that* $\tau_k \to \infty$ *as* $k \to \infty$ *and* $(x(n+\tau_k))$ *converges uniformly on* $\mathbb{Z}_+$ *as* $k \to \infty$*.*

Theorem 2.9. [4] *Suppose that an asymptotically almost periodic function* $f(t)$ *differentiable and its derivative* $f'(t)$ *is also asymptotically almost periodic. Then* $f'(t) = p(t)' + q'(t)$*.*

3. Main result

For a given $r \geq 0$, let

$$C = \{\phi : [-r, 0] \subset \mathbb{R} \to \mathbb{R}^n : \ \phi \text{ is continuous}\}$$

with the norm

$$|\phi| = \sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}, \phi \in C.$$

We denote by C_α the set of ϕ such that $|\phi| < \alpha$ and $\overline{C}_\alpha$ is the closure of C_α. Consider a system of functional differential equations

$$x'(t) = f(t, x_t), \tag{3.1}$$

where x_t denotes the function $x(t+\theta), -r \leq \theta \leq 0$, that is, $x_t \in C$. For the definition of the solution and the existence theorem, see [1]. We denote by $x(t) = x(t, t_0, \phi)$ a solution of (3.1) such that $x_{t_0}(t_0, \phi) = \phi$. Let $f : \mathbb{R} \times \overline{C}_B \to \mathbb{R}^n$ be a continuous function. We assume that $f(t, \phi)$ is uniformly almost periodic.

Yoshizawa [3] obtained the following existence theorem.

THEOREM 3.1. *Suppose that the system (3.1) has a solution $x(t)$ defined on $\mathbb{R}_+$ such that $|x_t| \leq B$ for $t \geq 0$. If $x(t)$ is asymptotically almost periodic, then the system (3.1) has an almost periodic solution.*

This theorem can be proven by Theorems 2.6 and 2.9. For the detail, see [3, Theorem 1].

For a given positive integer $r > 0$, we define C to be a Banach space with the norm $|\cdot|$ by

$$C = \{\phi : [-r, 0] \to \mathbb{R}^n : |\phi| = \max_{-r \leq \tau \leq 0} |\phi(\tau)|\},$$

where $[-r, 0]$ is a discrete interval. Let $n_0 \in \mathbb{Z}_+$ and let $x = (x(n))_{n \geq n_0 - r}$ be a sequence in $\mathbb{R}^n$. For each $n \geq n_0$, we define $x_n : [-r, 0] \to \mathbb{R}^n$ by

$$x_n(\tau) = x(n+\tau),\ \tau \in [-r, 0].$$

We consider the functional difference equation

$$x(n+1) = f(n, x_n), \tag{3.2}$$

where $f : \mathbb{Z} \times C_B \to \mathbb{R}^n$ and $x_n : [-r, 0] \to C$.

Now, we can obtain the following Theorem 3.3 as a discrete analogue of Theorem 3.1. To do this we need the following discretization of Theorem 2.6.

THEOREM 3.2. *Let $f : \mathbb{Z} \times D \to \mathbb{R}^n$ be continuous in $x \in D$ for each $n \in \mathbb{Z}$, where $D \subset \mathbb{R}^n$ is an open set. Assume that $f(n, x)$ is uniformly almost periodic and $p : \mathbb{Z} \to \mathbb{R}^n$ is an almost periodic sequence such that $p_n \in S \subset D$ for all $n \in \mathbb{Z}$, where S is a compact set in D and $p_n(\tau) = p(\tau + n)$ for $\tau \in [-r, 0]$. Then $f(n, p_n)$ is uniformly almost periodic.*

Proof. Let (τ'_k) be any sequence in $\mathbb{Z}$. Then, from Theorem 2.4, there exist a subsequence (τ_k) of (τ'_k) and a continuous and uniformly almost periodic function $g(n, \phi)$ such that

$$f(n + \tau_k, \phi) \to g(n, \phi)$$

uniformly on $\mathbb{Z} \times S$ as $k \to \infty$. Also, by Theorem 2.5, we have

$$p(n + \tau_k, \phi) \to \xi(n)$$

uniformly on $\mathbb{Z}$ as $k \to \infty$ for some almost periodic sequence $\xi \in S$. In view of continuity of $g(n, \phi)$, there exists a $\delta = \delta(\frac{\varepsilon}{2}) > 0$ such that when $|\phi - \psi| < \delta$,

$$|g(n, \phi) - g(n, \psi)| < \frac{\varepsilon}{2}, \; n \in \mathbb{Z}, \phi, \psi \in S.$$

Then there exists a $k_0 = k_0(\varepsilon) > 0$ such that for all $k \geq k_0$,

$$|f(n + \tau_k, \phi) - g(n, \phi)| < \frac{\varepsilon}{2}, \; n \in \mathbb{Z}, \phi \in S$$

and

$$|p(n + \tau_k, \phi) - \xi(n)| < \delta, \; n \in \mathbb{Z}.$$

Thus we obtain that for all $n \in \mathbb{Z}$,

$$|p_{n+\tau_k} - \xi_n| = \sup_{-r \leq \tau \leq 0} |p(n + \tau_k + \tau) - \xi(n + \tau)| < \delta(\frac{\epsilon}{2})$$

whenever $k \geq k_0$. Then we have

$$\begin{aligned} &|f(n + \tau_k, p_{n+\tau_k}) - g(n, \xi_n)| \\ &\leq |f(n + \tau_k, p_{n+\tau_k}) - g(n, p_{n+\tau_k})| + |g(n + \tau_k, p_{n+\tau_k}) - g(n, \xi_n)| \\ &< \varepsilon \end{aligned}$$

since $p_{n+\tau_k}, \xi_n \in S$. Therefore

$$f(n + \tau_k, p_{n+\tau_k}) \to g(n, \xi_n)$$

uniformly on $\mathbb{Z} \times S$ as $k \to \infty$. If follows from Theorem 2.4 that $f(n, p_n)$ is uniformly almost periodic. □

THEOREM 3.3. *Suppose that the system (3.2) has a bounded solution $x(n)$ on $\mathbb{Z}_+$ with $|x_n| \leq B^*, B^* < B$, for all $n \in \mathbb{Z}_+$. If $x(n)$ is asymptotically almost periodic, then (3.2) has an almost periodic solution.*

Proof. Since $x(n)$ is asymptotically almost periodic, $x(n) = p(n) + q(n)$, where $p(n)$ is an almost periodic sequence and $\lim_{n \to \infty} q(n) = 0$. Since $p(n)$ is almost periodic, we have

$$p(n + \tau_k) \to p(n)$$

uniformly on $\mathbb{Z}$ as $k \to \infty$ for some $(\tau_k) \subset \mathbb{Z}$ with $\tau_k \to \infty$ as $k \to \infty$, by Theorem 2.5. Note that

$$x(n + \tau_k) = p(n + \tau_k) + q(n + \tau_k), \; n + \tau_k \geq 0.$$

Thus

$$\begin{aligned} |p(n + \tau_k)| &\leq |x(n + \tau_k)| + |q(n + \tau_k)| \\ &\leq B^* + |q(n + \tau_k)|, \; n + \tau_k \geq 0. \end{aligned}$$

By letting $k \to \infty$, we have $|p(n)| \le B^*, n \in \mathbb{Z}$. In view of Theorem 3.2, $f(n, p_n)$ is uniformly almost periodic.

Now, we show that $p(n+1) = f(n, p_n), n \in \mathbb{Z}$, i.e., $p(n)$ is a solution of (3.2). Since $x(n)$ is a solution of (3.2), we have

$$\begin{aligned} x(n+1) &= f(n, x_n) \\ &= f(n, p_n) + f(n, x_n) - f(n, p_n), \ n \in \mathbb{Z}. \end{aligned}$$

Also, we have

$$x(n+1) = p(n+1) + q(n+1) \to p(n+1) \text{ as } n \to \infty.$$

Since $f(n, x_n) - f(n, p_n) \to 0$ as $n \to \infty$, we obtain that

$$p(n+1) = f(n, p_n), n \in \mathbb{Z}.$$

This completes the proof. □

Song [2] obtained the following result as a discrete analogue of Yoshizawa's result in [3].

THEOREM 3.4. *Suppose that (3.2) has a bounded solution $x(n)$ on $\mathbb{Z}_+$. If $x(n)$ is uniformly stable, then $x(n)$ is asymptotically almost periodic. Thus (3.2) has an asymptotically almost periodic solution. For the definition of uniform stability, see* [2].

COROLLARY 3.5. *Under the same conditions in Theorem 3.4, (3.2) has an almost periodic solution.*

Proof. From Theorem 3.4, the solution $x(n)$ is asymptotically almost periodic. Thus $x(n)$ is an almost periodic solution of (3.2) by Theorem 3.3. □

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