JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 20, No. 4, December 2007

# CHARACTERIZATIONS OF GAMMA DISTRIBUTION

Min-Young Lee* and Eun-Hyuk Lim**

Abstract. Let $X_1, \cdots, X_n$ be nondegenerate and positive independent identically distributed(i.i.d.) random variables with common absolutely continuous distribution function $F(x)$ and $E(X^2) < \infty$. The random variables $X_1 + \cdots + X_n$ and $\frac{X_1+\cdots+X_m}{X_1+\cdots+X_n}$ are independent for $1 \leq m < n$ if and only if $X_1, \cdots, X_n$ have gamma distribution.

## 1. Introduction

A random variable $X$ has a gamma distribution with shape parameter $\lambda$ and scale parameter $\alpha$, if its p.d.f. is given by

$$f(x; \alpha, \lambda) = \begin{cases} 0, & x < 0, \\ \frac{\alpha^\lambda}{\Gamma(\lambda)} x^{\lambda-1} e^{-\alpha x}, & x \geq 0. \end{cases}$$

The characteristic function of the distribution is given by

$$\phi\ (t; \alpha, \lambda) = (1 - \frac{it}{\alpha})^{-\lambda}.$$

Here $\alpha$, $\lambda > 0$ are two parameters.

Characterizations of the gamma distribution have been extensively studied in the literature. Lukacs(1955) obtained the following nice characterization of the gamma distribution that if random variables $X$, $Y$ are independent non-degenerate and positive then the random variables $\frac{X}{Y}$ and $X + Y$ are independent if and only if $X$ and $Y$ have gamma distributions. This result has been the starting point of numerous investigations. Kotz(1974)

The present research was conducted by the research fund of Dankook University in 2006.

Received September 28, 2007.

2000 *Mathematics Subject Classifications*: 60E10, 62E15.

Key words and phrases: independent identically distributed, a statistic scale-invariant, gamma distribution.

shows a good review of the gamma distribution including general prperties and characterizations. Recently, Ghitany, El-saidi and Khalil(1995) obtain a characterization of gamma distribution for a general class of life-testing models based on relationship between conditional expectation and the failure rate function.

In this paper, we extend the theorem of Lukacs and obtain the following result that the random variables $X_1 + \cdots + X_n$ and $\frac{X_1+\cdots+X_m}{X_1+\cdots+X_n}$ are independent for $1 \leq m < n$ if and only if $X_1, \cdots, X_n$ have gamma distribution.

## 2. Main results

THEOREM 1. *Let $X$, $Y$ and $Z$ be nondegenerate and positive i.i.d. random variables with common absolutely continuous distribution function $F(x)$ and $E(X^2) < \infty$. The random variables $X+Y+Z$ and $\frac{X+Y}{X+Y+Z}$ are independent if and only if $X$, $Y$ and $Z$ have gamma distribution.*

*Proof.* Since $\frac{X+Y}{X+Y+Z}$ is a statistic scale-invariant, $\frac{X+Y}{X+Y+Z}$ and $X+Y+Z$ are independent for gamma variable[see Lukacs and Laha(1963)]. We have to prove the reverse.

We denote the characteristic functions of $X+Y+Z$ , $\frac{X+Y}{X+Y+Z}$ and $\left(X+Y+Z, \frac{X+Y}{X+Y+Z}\right)$ by $\phi_1(t)$, $\phi_2(s)$ and $\phi(t,s)$, respectively. The independence of $X+Y+Z$ and $\frac{X+Y}{X+Y+Z}$ is equivalent to

$$\phi(t,s) = \phi_1(t) \cdot \phi_2(s). \tag{1}$$

The left hand side of (1) becomes

$$\phi(t,s) = \int_0^\infty \int_0^\infty \int_0^\infty \exp\left\{it(x+y+z) + \frac{is(x+y)}{x+y+z}\right\} dF(x)dF(y)dF(z).$$

Also the right hand side of (1) becomes

$$\phi_1(t)\cdot\phi_2(s)=\int_0^\infty\int_0^\infty\int_0^\infty \exp\{it(x+y+z)\}dF(x)dF(y)dF(z)$$
$$\cdot\int_0^\infty\int_0^\infty\int_0^\infty \exp\left\{\frac{is(x+y)}{x+y+z}\right\}dF(x)dF(y)dF(z).$$

Then (1) gives

$$\begin{aligned}(2)\qquad &\int_0^\infty\int_0^\infty\int_0^\infty \exp\left\{it(x+y+z)+\frac{is(x+y)}{x+y+z}\right\}dF(x)dF(y)dF(z)\\ &=\int_0^\infty\int_0^\infty\int_0^\infty \exp\{it(x+y+z)\}dF(x)dF(y)dF(z)\\ &\qquad\cdot\int_0^\infty\int_0^\infty\int_0^\infty \exp\left\{\frac{is(x+y)}{x+y+z}\right\}dF(x)dF(y)dF(z).\end{aligned}$$

The integrals in (2) exist not only for reals $t$ and $s$ but also for complex values $t=u+iv, s=u^*+iv^*$, where $u$ and $u^*$ are reals, for which $v=Im(t)\geq 0$, $v^*=Im(s)\geq 0$ and they are analytic for all $t$, $s$ for $v=Im(t)>0$, $v^*=Im(s)>0$, [see Lukacs(1955)].

Differentiating (2) twice, first with respect to $t$ and then respect to $s$ and setting $s=0$, we get

$$\begin{aligned}(3)\qquad &\int_0^\infty\int_0^\infty\int_0^\infty (x+y)^2\exp\{it(x+y+z)\}dF(x)df(y)dF(z)\\ &=\theta\int_0^\infty\int_0^\infty\int_0^\infty (x+y+z)^2\exp\{it(x+y+z)\}dF(x)dF(y)dF(z),\end{aligned}$$

where $\theta=E\left[\left(\frac{X+Y}{X+Y+Z}\right)^2\right]$.

The random variable $\theta$ is bounded. Therefore all its moments exist.

Note that

$\theta=E\left[\left(\frac{X+Y}{X+Y+Z}\right)^2\right]=E\left[\left(\frac{Y+Z}{X+Y+Z}\right)^2\right]=E\left[\left(\frac{X+Z}{X+Y+Z}\right)^2\right]$

for i.i.d. random variables $X$, $Y$ and $Z$. Then

$$
\begin{aligned}
3\theta &= E\left[\frac{(X+Y+Z)^2+X^2+Y^2+Z^2}{(X+Y+Z)^2}\right] \\
&= E\left[1+\frac{1}{1+\frac{2(XY+YZ+ZX)}{X^2+Y^2+Z^2}}\right].
\end{aligned} \tag{4}
$$

Note that, for $x>0$, $y>0$ and $z>0$, $0<2(xy+yz+zx)\le 2(x^2+y^2+z^2)$ and the equality on the right hand side occurs only if $x=y=z$. By the assumed continuity of $F(x)$, $P(x=y=z)=0$, so $0<\frac{2(xy+yz+zx)}{x^2+y^2+z^2}<2$, that is, by (4), $\frac{4}{9}<\theta<\frac{2}{3}$.

Let $\varphi(t)$ be the characteristic function of $F(x)$. Then

$$
\varphi'(t)=i\int_0^\infty x\exp\left[itx\right]dF(x)
$$

and

$$
\varphi''(t)=-\int_0^\infty x^2\exp\left[itx\right]dF(x).
$$

We can express (3) as a differential equation for the characteristic function $\varphi(t)$ and get

$$
2\varphi''(t)(\varphi(t))^2+2(\varphi'(t))^2\varphi(t)=\theta\{3\varphi''(t)(\varphi(t))^2+6(\varphi'(t))^2\varphi(t)\}.
$$

That is,

$$
\frac{\varphi''(t)}{\varphi'(t)}=\frac{6\theta-2}{2-3\theta}\frac{\varphi'(t)}{\varphi(t)},\qquad \frac{4}{9}<\theta<\frac{2}{3}.
$$

After integrating with the initial conditions $\varphi(0)=1$, $\varphi'(0)=iE(X)$, we get

$$
\varphi'(t)=iE(X)(\varphi(t))^{\frac{6\theta-2}{2-3\theta}},\quad \frac{6\theta-2}{2-3\theta}>1. \tag{5}
$$

The solution of this differential equation (5) with the above initial conditions is

$$
\varphi(t)=\left(1-\frac{iE(X)}{\lambda}t\right)^{-\lambda},\qquad \lambda=\frac{2-3\theta}{9\theta-4}>0.
$$

Therefore $F(x)$ is a gamma distribution.

THEOREM 2. *Let $X_1, \cdots, X_n$ be nondegenerate and positive i.i.d. random variables with common absolutely continuous distribution function $F(x)$ and $E(X^2) < \infty$. The random variables $X_1 + \cdots + X_n$ and $\frac{X_1+\cdots+X_m}{X_1+\cdots+X_n}$ are independent for $1 \le m < n$ if and only if $X_1, \cdots, X_n$ have gamma distribution.*

*Proof.* Since $\frac{X_1+\cdots+X_m}{X_1+\cdots+X_n}$ is a statistic scale-invariant, $\frac{X_1+\cdots+X_m}{X_1+\cdots+X_n}$ and $X_1 + \cdots + X_m$ are independent for gamma variable [see Lukacs and Laha(1963)]. We have to prove the converse.

We denote the characteristic funtions of $X_1 + \cdots + X_n$ , $\frac{X_1+\cdots+X_m}{X_1+\cdots+X_n}$ and $\left(X_1 + \cdots + X_n, \frac{X_1+\cdots+X_m}{X_1+\cdots+X_n}\right)$ by $\phi_1(t)$, $\phi_2(s)$ and $\phi(t,s)$, respectively. The independence of $X_1 + \cdots + X_n$ and $\frac{X_1+\cdots+X_m}{X_1+\cdots+X_n}$ is equivalent to

$$\phi(t,s) = \phi_1(t) \cdot \phi_2(s). \tag{6}$$

The left hand side of (6) becomes

$$\phi(t,s) = \int_0^\infty \cdots \int_0^\infty \exp\left[it(x_1 + \cdots + x_n) + \frac{is(x_1 + \cdots + x_m)}{x_1 + \cdots + x_n}\right] dF(x_1) \cdots dF(x_n).$$

Also the right hand side of (6) becomes

$$\phi_1(t) \cdot \phi_2(s) = \int_0^\infty \cdots \int_0^\infty \exp\left[it(x_1 + \cdots + x_n)\right] dF(x_1) \cdots dF(x_n) \cdot \int_0^\infty \cdots \int_0^\infty \exp\left[\frac{is(x_1 + \cdots + x_m)}{x_1 + \cdots + x_n}\right] dF(x_1) \cdots dF(x_n).$$

Then (6) gives

$$\begin{aligned} &\int_0^\infty \cdots \int_0^\infty \exp\left[it(x_1 + \cdots + x_n) + \frac{is(x_1 + \cdots + x_m)}{x_1 + \cdots + x_n}\right] dF(x_1) \cdots dF(x_n) \\ &= \int_0^\infty \cdots \int_0^\infty \exp\left[it(x_1 + \cdots + x_n)\right] dF(x_1) \cdots dF(x_n) \\ &\quad \cdot \int_0^\infty \cdots \int_0^\infty \exp\left[\frac{is(x_1 + \cdots + x_m)}{x_1 + \cdots + x_n}\right] dF(x_1) \cdots dF(x_n). \end{aligned} \tag{7}$$

The integrals in (7) exist not only for reals $t$ and $s$ but also for complex values $t = u + iv, s = u^* + iv^*$, where $u$ and $u^*$ are reals, for which $v = Im(t) \geq 0$, $v^* = Im(s) \geq 0$ and they are analytic for all $t$, $s$ for $v = Im(t) > 0$, $v^* = Im(s) > 0$, [see Lukacs(1955)].

Differentiating (7) twice, first with respect to $t$ and then respect to $s$ and setting $s = 0$, we get

$$\begin{aligned} &\int_0^\infty \cdots \int_0^\infty (x_1 + \cdots + x_m)^2 \exp\{it(x_1 + \cdots + x_n)\} dF(x_1) \cdots dF(x_n) \\ (8) \qquad &= \theta \int_0^\infty \cdots \int_0^\infty (x_1 + \cdots + x_n)^2 \\ &\qquad \cdot \exp\{it(x_1 + \cdots + x_n)\} dF(x_1) \cdots dF(x_n), \end{aligned}$$

where $\theta = E\left[\left(\frac{X_1 + \cdots + X_m}{X_1 + \cdots + X_n}\right)^2\right]$.

The random variable $\theta$ is bounded. Therefore all its moments exist. By the property of expectation and the same way of proof of Theorem1, we can compute that Note that

$$\begin{aligned} {}_nC_m \theta &= E\left[\frac{\frac{{}_nC_m \cdot m}{n} \sum_{k=1}^n X_k^2 + \frac{{}_nC_m \cdot {}_mC_2}{{}_nC_2} 2 \sum\sum_{1 \leq i < j \leq n} X_i X_j}{(\sum_{k=1}^n X_k)^2}\right] \\ (9) \qquad &= E\left[\frac{{}_{n-2}C_{m-2} \cdot (\sum_{k=1}^n X_k)^2 + {}_{n-2}C_{m-1} \cdot \sum_{k=1}^n {X_k}^2}{(\sum_{k=1}^n X_k)^2}\right] \\ &= E\left[{}_{n-2}C_{m-2} + {}_{n-2}C_{m-1} \cdot \frac{1}{1 + \frac{2\sum\sum_{1 \leq i < j \leq n} X_i X_j}{\sum_{k=1}^n X_k^2}}\right]. \end{aligned}$$

Note that, for $x_1, \cdots, x_n > 0$, $0 < 2\sum_{1 \leq i < j \leq n} x_i x_j \leq (n-1)(x_1^2 + \cdots + x_n^2)$ and the equality on the right hand side occurs only if $x_1 = \cdots = x_n$. By the assumed continuity of $F(x)$, $P(x_1 = \cdots = x_n) = 0$, so $0 < \frac{2\sum_{1 \leq i < j \leq n} x_i x_j}{x_1^2 + \cdots + x_n^2} < n - 1$, that is, by (9), $\left(\frac{m}{n}\right)^2 < \theta < \frac{m}{n}$.

Let $\varphi(t)$ be the characteristic function of F(x). Then

$$\varphi'(t) = i \int_0^\infty x \exp[itx] dF(x)$$

and

$$\varphi''(t) = -\int_0^\infty x^2 \exp[itx] dF(x).$$

We can express (8) as a differential equation for the characteristic function $\varphi(t)$ and get

$$\begin{aligned}(m)\varphi''(t)(\varphi(t))^{n-1} &+ 2\ {}_mC_2(\varphi'(t))^2(\varphi(t))^{n-2} \\ &= \theta\{n\varphi''(t)(\varphi(t))^{n-1} + 2\ {}_nC_2(\varphi'(t))^2(\varphi(t))^{n-2}\}.\end{aligned}$$

That is,

$$\frac{\varphi''(t)}{\varphi'(t)} = \frac{2({}_nC_2\theta - {}_mC_2)}{m - n\theta}\frac{\varphi'(t)}{\varphi(t)}, \quad \left(\frac{m}{n}\right)^2 < \theta < \frac{m}{n}.$$

After integrating with the initial conditions $\varphi(0) = 1$, $\varphi'(0) = iE(X)$, we get

$$\varphi'(t) = iE(X)(\varphi(t))^{\frac{n(n-1)\theta - m(m-1)}{m-n\theta}}, \quad \frac{n(n-1)\theta - m(m-1)}{m - n\theta} > 1. \tag{10}$$

The solution of this differential equation (10) with the above initial conditions is

$$\varphi(t) = \left(1 - \frac{iE(X)}{\lambda}t\right)^{-\lambda}, \quad \lambda = \frac{m - n\theta}{n^2\theta - m^2} > 0.$$

Therefore $F(x)$ is a gamma distribution.

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Department of Mathematics
Dankook University
Cheonan Chungnam 330-714, Republic of Korea
*E-mail*: leemy@dankook.ac.kr

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Department of Mathematics
Dankook University
Cheonan Chungnam 330-714, Republic of Korea
*E-mail*: ehlim@dankook.ac.kr