JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **20**, No. 3, September 2007

DISCRETE VOLTERRA EQUATIONS IN WEIGHTED SPACES

Yoon Hoe Goo * and Dong Man Im**

Abstract. We prove the Medina's results about the existence and uniqueness of solutions of discrete Volterra equations of convolution type in weighted spaces, by using the well-known Contraction Mapping Principle.

1. Introduction

Discrete Volterra equations have been studied as the discrete analogue of Volterra integrodifferential equations. Volterra equation of convolution type

$$x(n+1) = Ax(n) + \sum_{j=0}^{n} B(n-j)x(j), \tag{1.1}$$

where A is a $k \times k$ real matrix and $B(n)$ is a $k \times k$ real matrix defined on $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$, appeared in [1] and [3].

In this paper we deal with the discrete convolution equation

$$x(n) = g(n) + \sum_{i=n_0}^{n-1} a(n-i)f(i, x(i)). \tag{1.2}$$

This equation is a special case of the retarded equation whose delay is infinite

$$x(n) = g(n) + \sum_{i=-\infty}^{n-1} f(n, i, x(i)) \tag{1.3}$$

Received August 6, 2007.
2000 Mathematics Subject Classification: Primary 39A12.
Key words and phrases: discrete Volterra equation, weighted space.

in [4]. Medina [4] studied the existence and uniqueness of solutions of

$$x(n) = g(n) + \sum_{i=n_0}^{n-1} f(n, i, x(i)), \quad n_0 \in \mathbb{Z} \tag{1.4}$$

in the weighted spaces.

Let $\omega : \mathbb{N}(n_0) \to (0, \infty)$, $\mathbb{N}(n_0) = \{n_0, n_0 + 1, \cdots\}$, $n_0 \in \mathbb{Z}$, be a weight function. Define the weighted space

$$S(\mathbb{N}(n_0), \omega) = \{x : \mathbb{N}(n_0) \to \mathbb{R} \mid |x|_\omega = \sup_{n \in \mathbb{N}(n_0)} \omega^{-1}(n)|x(n)| < \infty\}.$$

Then $S(\mathbb{N}(n_0), \omega)$ becomes a Banach space with the norm $|\cdot|_\omega$.

We prove in detail the results about the existence and uniqueness of solutions of (1.2) appeared in [4] without proof, by using the well-known Contraction Mapping Principle.

2. Main Results

For the discrete Volterra equation

$$x(n) = g(n) + \sum_{i=n_0}^{n-1} f(n, i, x(i)), \quad n_0 \in \mathbb{Z}, \quad n \in \mathbb{N}(n_0), \tag{2.1}$$

where $g : \mathbb{N}(n_0) \to \mathbb{R}$ and $f : \mathbb{N}(n_0) \times \mathbb{N}(n_0) \times \mathbb{R} \to \mathbb{R}$, Medina [4] showed the existence and uniqueness in the following :

Theorem 2.1. [4, Theorem 4] *For Eqn (2.1), assume that*
(i) there exists a function L such that, for $n_0 \le i \le n$ and $n \in \mathbb{N}(n_0)$,

$$|f(n, i, x) - f(n, i, y)| \le L(n, i)|x - y|, \; x, y \in S(\mathbb{N}(n_0), \omega),$$

(ii) $\sup_{n \in \mathbb{N}(n_0)} \left| \sum_{i=n_0}^{n-1} \omega^{-1}(n) f(n, i, 0) \right| < \infty$,
(iii) $0 < A = \sum_{i=n_0}^{\infty} sup_{n>i}[\omega^{-1}(n)\omega(i)L(n, i)] < 1$,
(iv) $g \in S(\mathbb{N}(n_0), \omega)$.
Then Eqn (2.1) has a unique solution.

We consider the discrete convolution equation

$$x(n) = g(n) + \sum_{i=n_0}^{n-1} a(n - i) f(n, x(i)), \quad n \in \mathbb{N}(n_0), \tag{2.2}$$

where $g : \mathbb{N}(n_0) \to \mathbb{R}$, $a : \mathbb{N}(n_0) \to \mathbb{R}$ and $f : \mathbb{N}(n_0) \times \mathbb{N}(n_0) \to \mathbb{R}$. As an application of Theorem 2.1, we obtain the following :

THEOREM 2.2. *For Eqn (2.2), we assume that*
(i) there exists $\lambda : \mathbb{N}(n_0) \to \mathbb{R}$, *such that*

$$|f(i,x) - f(i,y)| \le \lambda(i)|x-y|, \ \ x, y \in S(\mathbb{Z}, \omega),$$

(ii) $\sup_{n\in\mathbb{Z}} \left\{ \sum_{i=n_0}^{n-1} \omega^{-1}(n)|a(n-i)||f(i,0)| \right\} \le M < \infty,$
(iii) $0 < A = \sum_{i=n_0}^{\infty} \lambda(i) sup_{n>i} \omega^{-1}(n)\omega(i)|a(n-i)| < 1,$
(iv) $g \in S(\mathbb{Z}, \omega)$.
Then there exists a unique solution of Eqn (2.2).

Proof. Define $T : S(\mathbb{Z}, \omega) \to S(\mathbb{Z}, \omega)$ by

$$Tx(n) = g(n) + \sum_{i=n_0}^{n-1} a(n-i) f(i, x(i)), \quad n \in \mathbb{Z}.$$

We claim that T maps $S(\mathbb{Z}, \omega)$ into itself. Since $g \in S(\mathbb{Z}, \omega)$, it suffices to show that

$$|\sum_{i=n_0}^{n-1} a(n-i) f(i, x(i))|_\omega < \infty.$$

$$\begin{aligned}
&\left| \sum_{i=n_0}^{n-1} a(n-i) f(i, x(i)) \right|_\omega = \sup_{n>i} \omega^{-1}(n) \left| \sum_{i=n_0}^{n-1} a(n-i) f(i, x(i)) \right| \\
&\le \sup_{n>i} \omega^{-1}(n) \left| \sum_{i=n_0}^{n-1} a(n-i) \{[f(i, x(i)) - f(i,0)] + f(i,0)\} \right| \\
&\le \sup_{n>i} \omega^{-1}(n) \left| \sum_{i=n_0}^{n-1} \lambda(i) x + f(i,0) \right| \\
&\le \sup \left| \sum_{i=n_0}^{n-1} \omega^{-1}(n) \lambda(i) \omega^{-1}(i) a(n-i) x(i) \right| \\
&\qquad\qquad + \sup \left| \sum_{i=n_0}^{n-1} \omega^{-1}(n) |a(n-i)||f(i,0)| \right| \\
&\le \sum_{i=n_0}^{n-1} \sup_{n>i} \{\omega^{-1}(n) \lambda(i) a(n-i) |x|_\omega\} + M \\
&\le A|x|_\omega + M \\
&< \infty.
\end{aligned}$$

This implies that T maps $S(\mathbb{Z},\omega)$ into itself. Now, we show that T is a contraction. For $x, y \in S(\mathbb{Z},\omega)$, we have

$$\omega^{-1}(n)|x(n)-y(n)| = \sum_{i=n_0}^{n-1} \omega^{-1}(n)|a(n,i)[f(i,x(i)) - f(n,i,y(i))]|.$$

Thus we obtain

$$\begin{aligned} |Tx - Ty|_\omega &\leq \sum_{i=n_0}^{n-1} \sup_{n>i} \omega^{-1}(n)\omega(i)\lambda(i)\omega^{-1}(i)a(n-i)|x(i)-y(i)| \\ &\leq A|x-y|_\omega. \end{aligned}$$

From (iii), T is a contraction. Therefore, by the well-known Contraction Mapping Principle, there exists a unique fixed point $x \in S(\mathbb{Z},\omega)$ of T, and it is a unique solution of Eqn (2.2). This completes the proof. □

THEOREM 2.3. *Let the same assumptions as in Theorem 2.2. In addition, we assume that the weight function ω is monotone nondecreasing. Then Eqn (2.2) has a unique solution in $S(\mathbb{Z},\omega)$ whenever*

$$\sum_{i=n_0}^{\infty} \lambda(i) \sup_{n>i} |a(n-i)| < 1. \tag{2.3}$$

Proof. Since w is monotone nondecreasing, we have $\frac{\omega(i)}{\omega(n)} \leq 1$ when $i \leq n$. Then the condition (iii) in Theorem 2.2 becomes

$$\begin{aligned} \sum_{i=n_0}^{\infty} \lambda(i) \sup_{n>i} \frac{\omega(i)}{\omega(n)} |a(n-i)| &\leq \sum_{i=n_0}^{\infty} \lambda(i) \sup_{n>i} |a(n-i)| \\ &< 1. \end{aligned}$$

Hence, by Theorem 2.2, Eqn (2.2) has a unique solution. □

REMARK 2.4. The condition (2.3) in Theorem 2.3 can be simplified as

$$\sup_{n>i} \sum_{i=n_0}^{n-1} \lambda(i)|f(i,0)| < \infty \quad \text{and} \quad \sum_{i=n_0}^{\infty} \lambda(i) < 1$$

when $|a|$ is monotone increasing.

References

[1] S. K. Choi and N. J. Koo, Asymptotic equivalence between two linear volterra difference systems, Comput. Math. Appl. **47**(2004), 461-471.
[2] S, Elaydi, Stability of Volterra difference of convolution type, Proc. Special program at Nankai Inst of Math. World Scientific, Singapore (1993), 66-73.
[3] S, Elaydi, An Introduction to Difference Equations, third ed. Springer Science + Business Media, New York, 2005.
[4] R. Medina, Solvability of discrete Volterra equations in weighted spaces, Dyn. Sys. Appl. **5**(1996), 407-422.

*
Department of Mathematics
Hanseo University
Seosan, Chungnam, 356-706, Republic of Korea
E-mail: yhgoo@hanseo.ac.kr

**
Department of Mathematics Education
Cheongju University
Cheongju, 360-764, Republic of Korea
E-mail: dmim@cju.ac.kr