

LIFTING T -STRUCTURES AND THEIR DUALS

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ABSTRACT. We define and study a concept of T^f -space for a map, which is a generalized one of a T -space, in terms of the Gottlieb set for a map. We show that X is a T^f -space if and only if $G(\Sigma B; A, f, X) = [\Sigma B, X]$ for any space B . For a principal fibration $E_k \rightarrow X$ induced by $k : X \rightarrow X'$ from $\epsilon : PX' \rightarrow X'$, we obtain a sufficient condition to having a lifting T^f -structure on E_k of a T^f -structure on X . Also, we define and study a concept of co- T^g -space for a map, which is a dual one of T^f -space for a map. We obtain a dual result for a principal cofibration $i_r : X \rightarrow C_r$ induced by $r : X' \rightarrow X$ from $\iota : X' \rightarrow cX'$.

1. Introduction

In [1], Aguade introduced a T -space as a space X having the property that the evaluation fibration $\Omega X \rightarrow X^{S^1} \rightarrow X$ is fibre homotopically trivial. It is easy to show that any H -space is a T -space. However, there are many T -spaces which are not H -spaces in [16]. Let ΣX denotes the reduced suspension of X , and ΩX denotes the based loop space of X . Let τ be the adjoint functor from the group $[\Sigma X, Y]$ to the group $[X, \Omega Y]$. The symbols e and e' denote $\tau^{-1}(1_{\Omega X})$ and $\tau(1_{\Sigma X})$ respectively. In [16], Woo and Yoon showed that the concept of T -space is closely related by the Gottlieb set $G(A, X)$, which is the set of homotopy classes of cyclic maps from A to X as follows; X is a T -space if and only if $G(\Sigma B, X) = [\Sigma B, X]$ for any space B . Also, we introduced and showed [16] that a concept of co- T -space as a dual one of T -space, which is closely related by the dual Gottlieb set $DG(X, A)$ which is the set of homotopy classes of cocyclic maps from X to A as follows; X is a co- T -space if and only if $DG(X; \Omega B) = [X, \Omega B]$ for any space B . In [12], Oda introduced the concept of f -cyclic map as a generalization of that

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of cyclic map. We called [21] the set of all homotopy classes of f -cyclic maps from B to X as the Gottlieb set $G(B; A, f, X)$ for a map $f : A \rightarrow X$. In general, $G(B, X) \subset G(B; A, f, X) \subset [B, X]$ for any map $f : A \rightarrow X$ and any space B . However, it is known [19] that $G(S^5, S^5 \times S^5) \cong 2Z \oplus 2Z \neq G(S^5; S^5, i_1, S^5 \times S^5) \cong 2Z \oplus Z \neq [S^5, S^5 \times S^5] \cong Z \oplus Z$, where $i_1 : S^5 \rightarrow S^5 \times S^5$ is the inclusion. In [18], we introduced the set of all homotopy classes of g -cocyclic maps from X to B as the dual Gottlieb set $DG(X, g, A; B)$ for a map $g : X \rightarrow A$. In general, $DG(X, B) \subset DG(X, g, A; B) \subset [X, B]$ for any map $g : X \rightarrow A$ and any space B . We also showed [20] that $DG(S^n \times S^n, K(Z, n)) \neq DG(S^n \times S^n, p_1, S^n; K(Z, n)) \neq [S^n \times S^n, K(Z, n)]$ for all n , where $p_1 : S^n \times S^n \rightarrow S^n$ is the projection.

In this paper, we introduce a T^f -space for a map $f : A \rightarrow X$ as a space X having the property that $e : \Sigma\Omega X \rightarrow X$ is f -cyclic, that is, there is a T^f -structure $F : \Sigma\Omega X \times A \rightarrow X$ on X . We show that X is a T^f -space if and only if $G(\Sigma B; A, f, X) = [\Sigma B, X]$ for any space B . There is an example which is a T^f -space for a map $f : A \rightarrow X$, but not T -space. We can also obtain, from some properties of T^f -spaces, that for any $x \in \pi_n(S^2)$, $\alpha \in \pi_k(S^2)$, $[x, \alpha] = 0$ for all $n \geq 3$, $k \geq 1$. It is known [16] that if X dominates A and X is a T -space, then A is a T -space. This fact can be generalized as follows. If X is a T^i -space for a map $i : A \rightarrow X$ and $i : A \rightarrow X$ has a left homotopy inverse $r : X \rightarrow A$, then A is a T -space. Moreover, let $p_k : E_k \rightarrow X$ be a principal fibration induced by $k : X \rightarrow X'$ from $\epsilon : PX' \rightarrow X'$. Let $F : \Sigma\Omega X \times A \rightarrow X$ be a T^f -structure on X . When can we have a $T^{\bar{f}}$ -structure $\bar{F} : \Sigma\Omega E_k \times E_l \rightarrow E_k$ on E_k such that $p_k \bar{F} \sim F(\Sigma\Omega p_k \times p_l) : E_k \times E_l \rightarrow X$? We can obtain an answer of the above question as follows. If X is a T^f -space with T^f -structure $F : \Sigma\Omega X \times A \rightarrow X$ and X' is a $T^{f'}$ -space with $T^{f'}$ -structure $F' : \Sigma\Omega X' \times A' \rightarrow X'$ such that $kF \sim F'(\Sigma\Omega k \times l) : \Sigma\Omega X \times A \rightarrow X'$, then there exists a $T^{\bar{f}}$ -structure $\bar{F} : \Sigma\Omega E_k \times E_l \rightarrow E_k$ on E_k such that $p_k \bar{F} \sim F(\Sigma\Omega p_k \times p_l) : \Sigma\Omega E_k \times E_l \rightarrow X$. As a corollary, we can obtain a sufficient condition to be E_k a T -space when X and X' are T -spaces.

On the other hand, we introduce a dual one of the above concept, co- T^g -space for a map $g : X \rightarrow A$ as a space X having the property that $e' : X \rightarrow \Omega\Sigma X$ is a g -cocyclic, that is, there is a co- T^g -structure $\theta : X \rightarrow \Omega\Sigma X \vee A$. We show that X is a co- T^g -space if and only if $DG(X, g, A; \Omega B) = [X, \Omega B]$ for any space B . It is known [16] that if X dominates A and X is a co- T -space, then A is a co- T -space. This fact can be generalized as follows. If X is a co- T^r -space for a map $r : X \rightarrow A$ and $r : X \rightarrow A$ has a right homotopy inverse $i : A \rightarrow X$, then

A is a co- T -space. Moreover, let $i_r : X \rightarrow C_r$ be a principal cofibration induced by $r : X' \rightarrow X$ from $\iota : X' \rightarrow cX'$. Let $\theta : X \rightarrow \Omega\Sigma X \vee A$ be a co- T^g -structure on X . When can we have a co- $T^{\bar{g}}$ -structure on C_r such that $(\Omega\Sigma i_r \vee i_s)\theta \sim \bar{\theta}i_r : X \rightarrow \Omega\Sigma C_r \vee C_s$? Then we show that if X is a co- T^g -space with co- T^g -structure $\theta : X \rightarrow \Omega\Sigma X \vee A$ and X' is a co- $T^{g'}$ -space with co- $T^{g'}$ -structure $\theta' : X' \rightarrow \Omega\Sigma X' \vee A'$ such that $(\Omega\Sigma r \vee s)\theta' \sim \theta r : X' \rightarrow \Omega\Sigma X \vee A$, then there exists a co- $T^{\bar{g}}$ -structure $\bar{\theta} : C_r \rightarrow \Omega\Sigma C_r \vee C_s$ on C_r such that $(\Omega\Sigma i_r \vee i_s)\theta \sim \bar{\theta}i_r : X \rightarrow \Omega\Sigma C_r \vee C_s$. As a corollary, we can obtain a sufficient condition to be C_r a co- T -space when X and X' are co- T -spaces.

2. Lifting T^f -structures

Let $f : A \rightarrow X$ be a map. A based map $g : B \rightarrow X$ is called *f-cyclic* [12] if there is a map $\phi : B \times A \rightarrow X$ such that the diagram

$$\begin{array}{ccc} A \times B & \xrightarrow{\phi} & X \\ j \uparrow & & \nabla \uparrow \\ A \vee B & \xrightarrow{(f \vee g)} & X \vee X \end{array}$$

is homotopy commute, where $j : A \vee B \rightarrow A \times B$ is the inclusion and $\nabla : X \vee X \rightarrow X$ is the folding map. We call such a map ϕ an *associated map* of a f -cyclic map g . Clearly, g is f -cyclic iff f is g -cyclic. In the case $f = 1_X : X \rightarrow X$, a map $g : B \rightarrow X$ is called cyclic [15]. We denote the set of all homotopy classes of f -cyclic maps from B to X by $G(B; A, f, X)$ which is called the *Gottlieb set for a map $f : A \rightarrow X$* . In the case $f = 1_X : X \rightarrow X$, we called such a set $G(B; X, 1, X)$ as the *Gottlieb set*, denoted by $G(B; X)$. In particular, $G(S^n; A, f, X)$ will be denoted by $G_n(A, f, X)$. Gottlieb [3,4] introduced and studied the *evaluation subgroups* $G_n(X) = G_n(X, 1, X)$ of $\pi_n(X)$.

In general, $G(B; X) \subset G(B; A, f, X) \subset [B, X]$ for any map $f : A \rightarrow X$ and any space B . However, there is an example [19] such that $G(B, X) \neq G(B; A, f, X) \neq [B, X]$. Thus we know that for any map $f : A \rightarrow X$, any cyclic map $g : B \rightarrow X$ is f -cyclic, but the converse does not hold.

The next proposition is an immediate consequence from the definition.

PROPOSITION 2.1.

- (1) For any maps $f : A \rightarrow X$, $\theta : C \rightarrow A$ and any space B , $G(B; A, f, X) \subset G(B; C, f\theta, X)$.
- (2) $G(B, X) = G(B; X, 1_X, X) \subset G(B; A, f, X) \subset G(B; A, *, X) = [B, X]$ for any spaces X , A and B .
- (3) $G(B, X) = \cap \{G(B; A, f, X) | f : A \rightarrow X \text{ is a map and } A \text{ is a space}\}$.
- (4) If $h : C \rightarrow A$ is a homotopy equivalence, then $G(B; A, f, X) = G(B; C, fh, X)$.
- (5) For any map $k : X \rightarrow Y$, $k_\#(G(B; A, f, X)) \subset G(B; A, kf, Y)$.
- (6) For any map $k : X \rightarrow Y$, $k_\#(G(B, X)) \subset G(B; X, k, Y)$.
- (7) For any map $s : C \rightarrow B$, $s^\#(G(B; A, f, X)) \subset G(C; A, f, X)$.

The following proposition says that T -spaces are completely characterized by the Gottlieb sets.

PROPOSITION 2.2. [16] X is a T -space if and only if $G(\Sigma B, X) = [\Sigma B, X]$ for any space B .

Aguade showed [1] that X is a T -space if and only if $e : \Sigma\Omega X \rightarrow X$ is cyclic. Now, for a map $f : A \rightarrow X$, we would like to introduce new spaces which can be characterized by the Gottlieb sets for a map $f : A \rightarrow X$.

DEFINITION 2.3. A space X is called a T^f -space for a map $f : A \rightarrow X$ if there is a map, T^f -structure on X , $F : \Sigma\Omega X \times A \rightarrow X$ such that $Fj \sim \nabla(e \vee f)$, where $j : \Sigma\Omega X \vee A \rightarrow \Sigma\Omega X \times A$ is the inclusion.

Clearly, any T -space means a T^1 -space. A space X is called an H^f -space for a map $f : A \rightarrow X$ [20] if there is a map, H^f -structure on X , $F : X \times A \rightarrow X$ such that $Fj \sim \nabla(1 \vee f)$, where $j : X \vee A \rightarrow X \times A$ is the inclusion. We can easily show that any H^f -space for a map $f : A \rightarrow X$ is a T^f -space for a map $f : A \rightarrow X$ for we can take a T^f -structure $F' = F(e \times 1) : \Sigma\Omega X \times A \rightarrow X$, where $F : X \times A \rightarrow X$ is an H^f -structure on X .

The following theorem says that a T^f -space can be characterized by the Gottlieb sets for a map $f : A \rightarrow X$.

THEOREM 2.4. X is a T^f -space for a map $f : A \rightarrow X$ if and only if $G(\Sigma B; A, f, X) = [\Sigma B, X]$ for any space B .

Proof. Suppose that X is a T^f -space for a map $f : A \rightarrow X$. Then there is a map $F : \Sigma\Omega X \times A \rightarrow X$ such that $Fj \sim \nabla(e \vee f)$, where $j : \Sigma\Omega X \vee A \rightarrow \Sigma\Omega X \times A$ is the inclusion. Let $g \in [\Sigma B, X]$. Consider the map $G = F(\Sigma\tau(g) \times 1) : \Sigma B \times A \rightarrow X$. Then $Gj \sim \nabla(g \vee f)$ and $g \in G(\Sigma B; A, f, X)$. On the other hand, suppose that $G(\Sigma B; A, f, X) =$

$[\Sigma B, X]$ for any space B . Take $B = \Omega X$ and consider the map $e : \Sigma \Omega X \rightarrow X$. Since $e \in G(\Sigma \Omega X; A, f, X)$, we know that the map e is f -cyclic and X is a T^f -space for a map $f : A \rightarrow X$. $\square$

It is known [16] that if X dominates A and X is a T -space, then A is a T -space. This fact can be generalized as the following corollary.

COROLLARY 2.5. *Let X be a T^i -space for a map $i : A \rightarrow X$.*

- (1) *If $i : A \rightarrow X$ has a left homotopy inverse $r : X \rightarrow A$, then A is a T -space.*
- (2) *If $i : A \rightarrow X$ has a right homotopy inverse $r : X \rightarrow A$, then X is a T -space.*

Proof. (1) Let B be any space. It is sufficient to show that $[\Sigma B, A] \subset G(\Sigma B, A)$ for any space B . Since X is a T^i -space for $i : A \rightarrow X$, we know, from Theorem 2.4, that $G(\Sigma B; A, i, X) = [\Sigma B, X]$. Thus we have, from Proposition 2.1(5), that $[\Sigma B, A] = r_*[\Sigma B, X] = r_*(G(\Sigma B; A, i, X)) \subset G(\Sigma B; A, ri, A) = G(\Sigma B, A, 1, A) = G(\Sigma B, A)$. Thus A is a T -space. (2) We show that $[\Sigma B, X] \subset G(\Sigma B, X)$ for any space B . By Theorem 2.4 and Proposition 2.1(1), we can obtain that $[\Sigma B, X] = G(\Sigma B; A, i, X) \subset G(\Sigma B; X, ir, X) = G(\Sigma B; X, 1, X) = G(\Sigma B, X)$. Thus we know, from Proposition 2.2, that X is a T -space. $\square$

From Proposition 2.1(2),(3), Proposition 2.2 and Theorem 2.4, we have the following corollary.

COROLLARY 2.6. *X is a T -space if and only if for any space A and any map $f : A \rightarrow X$, X is a T^f -space for a map $f : A \rightarrow X$.*

DEFINITION 2.7. *For a map $f : A \rightarrow X$, $P(\Sigma B; A, f, X) = \{\alpha \in [\Sigma B, X] \mid [f_\#(\beta), \alpha] = 0 \text{ for any space } C \text{ and any map } \beta \in [\Sigma C, A]\}$. A space X is called a GW^f -space for a map $f : A \rightarrow X$ if for any space B , $P(\Sigma B; A, f, X) = [\Sigma B, X]$.*

PROPOSITION 2.8. *$G(\Sigma B; A, f, X) \subset P(\Sigma B; A, f, X)$ for any space B .*

Proof. Let $[h] \in G(\Sigma B; A, f, X)$. Then there is a map $H : A \times \Sigma B \rightarrow X$ such that $Hj \sim \nabla(f \vee h)$, where $j : A \vee \Sigma B \rightarrow A \times \Sigma B$ is the inclusion. Let C be a space and $\beta = [g] \in [\Sigma C, A]$. Then consider the map $F = H(g \times 1) : \Sigma C \times \Sigma B \xrightarrow{(g \times 1)} A \times \Sigma B \xrightarrow{H} X$. Then $Fj' \sim \nabla(fg \vee h)$, where $j' : \Sigma C \vee \Sigma B \rightarrow \Sigma C \times \Sigma B$ is the inclusion. Thus we have $[f_\#(\beta), [h]] = 0$ and $[h] \in P(\Sigma B; A, f, X)$. $\square$

COROLLARY 2.9. *If X is a T^f -space for a map $f : A \rightarrow X$, then X is a GW^f -space for $f : A \rightarrow X$.*

Consider the natural pairing $\mu : S^3/S^1 = S^2 \times S^3 \rightarrow S^3/S^1 = S^2$. Thus we know that the Hopf map $\eta : S^3 \rightarrow S^2$ is cyclic. Thus η is e -cyclic and e is η -cyclic, that is, S^2 is a T^η -space. Thus we know that S^2 is a GW^η -space for $\eta : S^3 \rightarrow S^2$. On the other hand, it is known [16] that H -spaces and T -spaces are equivalent in the category of spheres. Thus we know that S^2 is not a T -space. Moreover, it is known [14] that $\eta_\# : \pi_n(S^3) \rightarrow \pi_n(S^2)$, $\eta_\#(\beta) = \eta \circ \beta$, is an isomorphism for $n \geq 3$. Thus we have the following example.

EXAMPLE 2.10.

- (1) S^2 is a T^η -space, but not T -space.
- (2) For any $x \in \pi_n(S^2)$, $\alpha \in \pi_k(S^2)$ ($n \geq 3$, $k \geq 1$), $[x, \alpha] = 0$.

Let $f : A \rightarrow X$, $f' : A' \rightarrow X'$, $l : A \rightarrow A'$, $k : X \rightarrow X'$ be maps. Then a pair of maps $(k, l) : (X, A) \rightarrow (X', A')$ is called a *map from f to f'* if the following diagram is commutative;

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ l \downarrow & & \downarrow k \\ A' & \xrightarrow{f'} & X'. \end{array}$$

It will be denoted by $(k, l) : f \rightarrow f'$.

Given maps $f : A \rightarrow X$, $f' : A' \rightarrow X'$, let $(k, l) : f \rightarrow f'$ be a map from f to f' . Let PX' and PA' be the spaces of paths in X' and A' which begin at $*$ respectively. Let $\epsilon_{X'} : PX' \rightarrow X'$ and $\epsilon_{A'} : PA' \rightarrow A'$ be the fibrations given by evaluating a path at its end point. Let $p_k : E_k \rightarrow X$ be the fibration induced by $k : X \rightarrow X'$ from $\epsilon_{X'}$. Let $p_l : E_l \rightarrow A$ induced by $l : A \rightarrow A'$ from $\epsilon_{A'}$. Then there is a map $\bar{f} : E_l \rightarrow E_k$ such that the following diagram is commutative

$$\begin{array}{ccc} E_l & \xrightarrow{\bar{f}} & E_k \\ p_l \downarrow & & \downarrow p_k \\ A & \xrightarrow{f} & X, \end{array}$$

where $E_l = \{(a, \xi) \in A \times PA' | l(a) = \epsilon(\xi)\}$, $E_k = \{(x, \eta) \in X \times PX' | k(x) = \epsilon(\eta)\}$, $\bar{f}(a, \xi) = (f(a), f' \circ \xi)$, $p_k(x, \eta) = x$, $p_l(a, \xi) = a$.

DEFINITION 2.11. *Let X be a T^f -space with T^f -structure $F : \Sigma\Omega X \times A \rightarrow X$. A map $(k, l) : f \rightarrow f'$ is called a T^f -primitive with respect to*

F if there is an associate map $F' : \Sigma\Omega X' \times A' \rightarrow X'$ of $e_{X'}$ -cyclic map f' such that the following diagram is homotopy commutative;

$$\begin{array}{ccc} \Sigma\Omega X \times A & \xrightarrow{F} & X \\ \Sigma\Omega k \times l \downarrow & & k \downarrow \\ \Sigma\Omega X' \times A' & \xrightarrow{F'} & X'. \end{array}$$

The following lemmas are standard.

LEMMA 2.12. A map $l : C \rightarrow X$ can be lifted to a map $C \rightarrow E_k$ if and only if $kl \sim *$.

LEMMA 2.13. [5] Given maps $g_i : A_i \rightarrow E_k$, $i = 1, 2$ and $g : A_1 \times A_2 \rightarrow E_k$ satisfying $p_k g|_{A_i} \sim p_k g_i$, $i = 1, 2$, then there is a map $h : A_1 \times A_2 \rightarrow E_k$ such that $p_k h = p_k g$ and $h|_{A_i} \sim g_i$, $i = 1, 2$.

THEOREM 2.14. If X is a T^f -space with T^f -structure $F : \Sigma\Omega X \times A \rightarrow X$ and $(k, l) : f \rightarrow f'$ is a T^f -primitive with respect to F , then there exists a $T^{\bar{f}}$ -structure $\bar{F} : \Sigma\Omega E_k \times E_l \rightarrow E_k$ on E_k such that the following diagram is homotopy commutative;

$$\begin{array}{ccc} \Sigma\Omega E_k \times E_l & \xrightarrow{\bar{F}} & E_k \\ \Sigma\Omega p_k \times p_l \downarrow & & p_k \downarrow \\ \Sigma\Omega X \times A & \xrightarrow{F} & X. \end{array}$$

Proof. Since $(k, l) : f \rightarrow f'$ is a T^f -primitive with respect to F , there is a map $F' : \Sigma\Omega X' \times A' \rightarrow X'$ such that $kF \sim F'(\Sigma\Omega k \times l) : \Sigma\Omega X \times A \rightarrow X'$. Then $kF(\Sigma\Omega p_k \times p_l) \sim F'(\Sigma\Omega k \times l)(\Sigma\Omega p_k \times p_l) = F'(\Sigma\Omega(k \circ p_k) \times l \circ p_l) \sim F'(* \times *) \sim * : \Sigma\Omega E_k \times E_l \rightarrow X'$. From Lemma 2.12, there is a lifting $\tilde{F} : \Sigma\Omega E_k \times E_l \rightarrow E_k$ of $F(\Sigma\Omega p_k \times p_l) : \Sigma\Omega E_k \times E_l \rightarrow X$, that is, $p_k \tilde{F} = F(\Sigma\Omega p_k \times p_l)$. Then $p_k \circ \tilde{F}|_{\Sigma\Omega E_k} \sim F|_{\Sigma\Omega X} \circ \Sigma\Omega p_k \sim p_k \circ e_{E_k}$ and $p_k \circ \tilde{F}|_{E_l} \sim F|_{A \circ p_l} \sim f \circ p_l = p_k \circ \bar{f}$. Thus we have, from Lemma 2.13, that there is a map $\bar{F} : \Sigma\Omega E_k \times E_l \rightarrow E_k$ such that $p_k \bar{F} = p_k \tilde{F} = F(\Sigma\Omega p_k \times p_l)$ and $\bar{F}|_{\Sigma\Omega E_k} \sim e_{E_k}$, $\bar{F}|_{E_l} \sim \bar{f}$. This proves the theorem. $\square$

Taking $f = 1_X$, $f' = 1_{X'}$ and $l = k$, we can obtain the following corollary.

COROLLARY 2.15. Let X and X' be T -spaces with T^1 -structures $E : \Sigma\Omega X \times X \rightarrow X$ and $E' : \Sigma\Omega X' \times X' \rightarrow X'$ respectively. If $k : X \rightarrow X'$ is a map satisfying $kE \sim E'(\Sigma\Omega k \times 1) : \Sigma\Omega X \times X \rightarrow X'$, then there is a T^1 -structure $\bar{E} : \Sigma\Omega E_k \times E_k \rightarrow E_k$ on E_k such that $p_k \bar{E} \sim E(\Sigma\Omega p_k \times p_k) : \Sigma\Omega E_k \times E_k \rightarrow X$.

In 1951, Postnikov [13] introduced the notion of the Postnikov system as follows; A *Postnikov system* for X (or *homotopy decomposition of X*) $\{X_n, i_n, p_n\}$ consists of a sequence of spaces and maps satisfying (1) $i_n : X \rightarrow X_n$ induces an isomorphism $(i_n)_\# : \pi_i(X) \rightarrow \pi_i(X_n)$ for $i \leq n$. (2) $p_n : X_n \rightarrow X_{n-1}$ is a fibration with fiber $K(\pi_n(X), n)$. (3) $p_n i_n \sim i_{n+1}$. It is well known fact [11] that if X is a 1-connected space having a homotopy type of CW-complex, then there is a Postnikov system $\{X_n, i_n, p_n\}$ for X such that $p_{n+1} : X_{n+1} \rightarrow X_n$ is the fibration induced from the path space fibration over $K(\pi_{n+1}(X), n+2)$ by a map $k^{n+2} : X_n \rightarrow K(\pi_{n+1}(X), n+2)$.

THEOREM 2.16. *Let A and X be spaces having the homotopy type of 1-connected countable CW-complexes, and $\{A_n, i'_n, p'_n\}$ and $\{X_n, i_n, p_n\}$ be Postnikov systems for A and X respectively. If X is a T^f -space with T^f -structure $F : \Sigma\Omega X \times A \rightarrow X$, then there exists a T^{f_n} -structure $F_n : \Sigma\Omega X_n \times A_n \rightarrow X_n$ for each stage X_n such that*

$$\begin{array}{ccc} \Sigma\Omega X_n \times A_n & \xrightarrow{F_n} & X_n \\ \Sigma\Omega p_n \times p'_n \downarrow & & \downarrow p_n \\ \Sigma\Omega X_{n-1} \times A_{n-1} & \xrightarrow{F_{n-1}} & X_{n-1}, \end{array}$$

where f_n is an induced map from f , and all the pair of k -invariants $(k_X^{n+2}, k_A^{n+2}) : f_n \rightarrow \tilde{f}_\#$ are T^{f_n} -primitive with respect to F_n , where $\tilde{f}_\# : K(\pi_{n+1}(A), n+2) \rightarrow K(\pi_{n+1}(X), n+2)$ is the induced map by $f : A \rightarrow X$.

Proof. Clearly $\{\Sigma\Omega X_n \times A_n, \Sigma\Omega i_n \times i'_n, \Sigma\Omega p_n \times p'_n\}$ is a Postnikov system for $\Sigma\Omega X \times A$. Then we have, by Kahn's result [7, Theorem 2.2], that there are families of maps $f_n : A_n \rightarrow X_n$ and $F_n : \Sigma\Omega X_n \times A_n \rightarrow X_n$ such that $p_n f_n = f_{n-1} p'_n$ and $i_n f \sim f_n i'_n$, and $p_n F_n = F_{n-1}(\Sigma\Omega p_n \times p'_n)$ and $i_n F \sim F_n(\Sigma\Omega i_n \times i'_n)$ for $n = 2, 3, \dots$ respectively, and $k_X^{n+2} f_n \sim \tilde{f}_\# k_A^{n+2} : A_n \rightarrow K(\pi_{n+1}(X), n+2)$ and $k_X^{n+2} F_n \sim \tilde{F}_\#(k_{\Sigma\Omega X}^{n+2} \times k_A^{n+2}) : X_n \times A_n \rightarrow K(\pi_{n+1}(X), n+2)$, where $k_A^{n+2} : A_n \rightarrow K(\pi_{n+1}(A), n+2)$, $k_X^{n+2} : X_n \rightarrow K(\pi_{n+1}(X), n+2)$ and $k_{\Sigma\Omega X}^{n+2} : \Sigma\Omega X_n \rightarrow K(\pi_{n+1}(\Sigma\Omega X), n+2)$ are k -invariants of A , X and $\Sigma\Omega X$ respectively, $\tilde{f}_\# : K(\pi_{n+1}(A), n+2) \rightarrow K(\pi_{n+1}(X), n+2)$ and $\tilde{F}_\# : K(\pi_{n+1}(\Sigma\Omega X), n+2) \times K(\pi_{n+1}(A), n+2) \approx K(\pi_{n+1}(\Sigma\Omega X \times A), n+2) \rightarrow K(\pi_{n+1}(X), n+2)$ are the induced maps by $f : A \rightarrow X$ and $F : X \times A \rightarrow X$ respectively. Since $F|_{\Sigma\Omega X} \sim e$ and $F_n|_{A_n} \sim f_n$, we know, from Kahn's another result [8, Theorem 1.2], that $F_n|_{\Sigma\Omega X_n} =$

$(F|_{\Sigma\Omega X})_n \sim e$ and $F_{n|A_n} = (F|_A)_n \sim f_n$. Thus there exists an T^{f_n} -structure $F_n : \Sigma\Omega X_n \times A_n \rightarrow X_n$ for each stage X_n such that

$$\begin{array}{ccc} \Sigma\Omega X_n \times A_n & \xrightarrow{F_n} & X_n \\ \Sigma\Omega p_n \times p'_n \downarrow & & \downarrow p_n \\ \Sigma\Omega X_{n-1} \times A_{n-1} & \xrightarrow{F_{n-1}} & X_{n-1}, \end{array}$$

where f_n is an induced map from f , and all the pair of k -invariants $(k_X^{n+2}, k_A^{n+2}) : f_n \rightarrow \tilde{f}_\#$ are T^{f_n} -primitive with respect to F_n , where $\tilde{f}_\# : K(\pi_{n+1}(A), n+2) \rightarrow K(\pi_{n+1}(X), n+2)$ is the induced map by $f : A \rightarrow X$. $\square$

In fact, the above theorem follows from Theorem 2.14 if we can show that all the pair of k -invariants $(k_X^{n+2}, k_A^{n+2}) : f_n \rightarrow \tilde{f}_\#$ are T^{f_n} -primitive with respect to F_n .

We can obtain an equivalent condition for E_k is a $T^{\bar{f}}$ -space for $\bar{f}$.

THEOREM 2.17. *Let $(k, l) : f \rightarrow f'$ be a map. Then E_k is a $T^{\bar{f}}$ -space for $\bar{f} : E_l \rightarrow E_k$ if and only if there is a map $G : \Sigma\Omega E_k \times E_l \rightarrow X$ such that $Gj \sim \nabla(p_k \circ e \vee p_k \circ \bar{f})$ and $kG \sim *$, where $j : \Sigma\Omega E_k \vee E_l \rightarrow \Sigma\Omega E_k \times E_l$ is the inclusion.*

Proof. Suppose that E_k is a $T^{\bar{f}}$ -space for $\bar{f} : E_l \rightarrow E_k$. Then there is a map $\bar{F} : \Sigma\Omega E_k \times E_l \rightarrow E_k$ such that $\bar{F}j' \sim \nabla(e \vee \bar{f})$. Let $G = p_k \bar{F} : \Sigma\Omega E_k \times E_l \rightarrow X$. Then $Gj \sim \nabla(p_k \circ e \vee p_k \circ \bar{f})$, where $j : \Sigma\Omega E_k \vee E_l \rightarrow \Sigma\Omega E_k \times E_l$ is the inclusion. Since G has a lifting $\bar{F}$, by Lemma 2.12, we know that $kG \sim *$. On the other hand, suppose there is a map $G : \Sigma\Omega E_k \times E_l \rightarrow X$ such that $Gj \sim \nabla(p_k \circ e \vee p_k \circ \bar{f})$ and $kG \sim *$, where $j : \Sigma\Omega E_k \vee E_l \rightarrow \Sigma\Omega E_k \times E_l$ is the inclusion. Since $kG \sim *$, there is a map $H : \Sigma\Omega E_k \times E_l \rightarrow E_k$ such that $p_k H \sim G$. For maps $e : \Sigma\Omega E_k \rightarrow E_k$ and $\bar{f} : E_l \rightarrow E_k$, we can easily know that $p_k H|_{\Sigma\Omega E_k} \sim p_k \circ e_{E_k}$ and $p_k H|_{E_l} \sim p_k \circ \bar{f}$. Thus we have, from Lemma 2.13, that there is a map $\bar{F} : \Sigma\Omega E_k \times E_l \rightarrow E_k$ such that $p_k \bar{F} = p_k H$ and $\bar{F}|_{\Sigma\Omega E_k} \sim e$ and $\bar{F}|_{E_l} \sim \bar{f}$. Thus we know that E_k is a $T^{\bar{f}}$ -space for $\bar{f} : E_l \rightarrow E_k$. $\square$

Now we can obtain the converse of Theorem 2.14 under some conditions as follows;

THEOREM 2.18. *Suppose that there are maps $s_k : X \rightarrow E_k$ and $s_l : A \rightarrow E_l$ such that $p_k s_k \sim 1_X$ and $p_l s_l \sim 1_A$. If there exists a $T^{\bar{f}}$ -structure $\bar{F} : \Sigma\Omega E_k \times E_l \rightarrow E_k$ on E_k such that the following diagram*

is homotopy commutative;

$$\begin{array}{ccc} \Sigma\Omega E_k \times E_l & \xrightarrow{\bar{F}} & E_k \\ \Sigma\Omega p_k \times p_l \downarrow & & \downarrow p_k \\ \Sigma\Omega X \times A & \xrightarrow{F} & X, \end{array}$$

then X is a T^f -space with T^f -structure $F : \Sigma\Omega X \times A \rightarrow X$.

Proof. Since E_k is a $T^{\bar{f}}$ -space for $\bar{f} : E_l \rightarrow E_k$, there is a map $G : \Sigma\Omega E_k \times E_l \rightarrow X$ such that $Gj \sim \nabla(p_k \circ e \vee p_k \circ \bar{f})$ and $kG \sim *$, where $j : \Sigma\Omega E_k \vee E_l \rightarrow \Sigma\Omega E_k \times E_l$ is the inclusion. Consider the map $F = G(\Sigma\Omega s_k \times s_l) : \Sigma\Omega X \times A \rightarrow X$. Then $Fj' \sim \nabla(e \vee f)$ and $kF(\Sigma\Omega p_k \times p_l) \sim *$, where $j' : \Sigma\Omega X \vee A \rightarrow \Sigma\Omega X \times A$ is the inclusion. Thus we know that X is a T^f -space with T^f -structure $F : \Sigma\Omega X \times A \rightarrow X$. $\square$

3. Extending co- T^g -structures

Let $g : X \rightarrow A$ be a map. A based map $f : X \rightarrow B$ is called *g-coclic* [12] if there is a map $\theta : X \rightarrow A \vee B$ such that the following diagram is homotopy commutative;

$$\begin{array}{ccc} X & \xrightarrow{\theta} & A \vee B \\ \Delta \downarrow & & \downarrow j \\ X \times X & \xrightarrow{(g \times f)} & A \times B, \end{array}$$

where $j : A \vee B \rightarrow A \times B$ is the inclusion and $\Delta : X \rightarrow X \times X$ is the diagonal map. We call such a map θ a *coassociated map* of a *g-cocyclic* map f .

In the case $g = 1_X : X \rightarrow X$, $f : X \rightarrow B$ is called *cocyclic* [15]. Clearly any cocyclic map is a *g-cocyclic* map and also $f : X \rightarrow B$ is *g-cocyclic* iff $g : X \rightarrow A$ is *f-cocyclic*. The *dual Gottlieb set* $DG(X, g, A; B)$ for a map $g : X \rightarrow A$ is the set of all homotopy classes of *g-cocyclic* maps from X to B . In the case $g = 1_X : X \rightarrow X$, we called such a set $DG(X, 1, X; B)$ as the *dual Gottlieb set*, denoted by $DG(X; B)$, that is, the dual Gottlieb set is exactly same with the dual Gottlieb set for the identity map. In particular, $DG(X, g, A; K(\pi, n))$ will be denoted by $G^n(X, g, A; \pi)$. Haslam [5] introduced and studied the *coevaluation subgroups* $G^n(X; \pi)$ of $H^n(X; \pi)$. $G^n(X; \pi)$ is defined to be the set of all homotopy classes of cocyclic maps from X to $K(\pi, n)$.

In general, $DG(X; B) \subset DG(X, g, A; B) \subset [X, B]$ for any map $g : X \rightarrow B$ and any space B . However, there is an example in [18] such that $DG(X, B) \neq DG(X, g, A; B) \neq [X, B]$.

The next proposition is an immediate consequence from the definition.

PROPOSITION 3.1.

- (1) For any maps $g : X \rightarrow A$, $h : A \rightarrow B$ and any space C , $DG(X, g, A; C) \subset DG(X, hg, B; C)$.
- (2) $DG(X, B) = DG(X, 1_X, X; B) \subset DG(X, g, A; B) \subset DG(X, *, A; B) = [X, B]$ for any spaces X, A and B .
- (3) $DG(X, B) = \cap \{DG(X, g, A; B) | g : X \rightarrow A \text{ is a map and } A \text{ is a space}\}$.
- (4) If $h : A \rightarrow B$ is a homotopy equivalence, then $DG(X, g, A; C) = DG(X, hg, B; C)$.
- (5) For any map $k : Y \rightarrow X$, $k^\#(DG(X, g, A; B)) \subset DG(Y, gk, A; B)$.
- (6) For any map $k : Y \rightarrow X$, $k^\#(DG(X; B)) \subset DG(Y, k, X; B)$.
- (7) For any map $s : B \rightarrow C$, $s_\#(DG(X, g, A; B)) \subset DG(X, g, A; C)$.

It is well known [5] that $G^n(X; \pi)$ is a subgroup of $H^n(X; \pi)$. Moreover, it is also shown [10] that if B is an H -group, then $DG(X, B)$ is a subgroup of $[X, B]$.

But we do not know whether $DG(X, g, A; B)$ is a group.

A space X is called a *co- T -space* [16] if $e' : X \rightarrow \Omega\Sigma X$ is cocyclic. The following proposition says that co- T -spaces are completely characterized by the dual Gottlieb sets.

PROPOSITION 3.2. [16] X is a co- T -space if and only if $DG(X, \Omega B) = [X, \Omega B]$ for any space B .

Now, for a map $g : X \rightarrow A$, we would like to introduce new spaces which can be characterized by the dual Gottlieb sets for a map $g : X \rightarrow A$.

DEFINITION 3.3. A space X is called a *co- T^g -space* for a map $g : X \rightarrow A$ if there is a map, a co- T^g -structure, $\theta : X \rightarrow \Omega\Sigma X \vee A$ such that $j\theta \sim (e' \times g)\Delta$, where $j : \Omega\Sigma X \vee A \rightarrow \Omega\Sigma X \times A$ is the inclusion and $\Delta : X \rightarrow X \times X$ is the diagonal map.

The following proposition says that co- T^g -spaces are completely characterized by the dual Gottlieb sets for a map $g : X \rightarrow A$.

PROPOSITION 3.4. [18] X is a co- T^g -space for a map $g : X \rightarrow A$ if and only if $DG(X, g, A; \Omega B) = [X, \Omega B]$ for any space B .

It is clear, from Proposition 3.1(2) and the above propositions, that any co- T -space is a co- T^g -space for any map $g : X \rightarrow A$. It is known [18] that if X dominates A and X is a co- T -space, then A is a co- T -space. This fact can be generalized as follows;

COROLLARY 3.5. *Let X be a co- T^r -space for a map $r : X \rightarrow A$.*

- (1) *If $r : X \rightarrow A$ has a right homotopy inverse $i : A \rightarrow X$, then A is a co- T -space.*
- (2) *If $r : X \rightarrow A$ has a left homotopy inverse $i : A \rightarrow X$, then X is a co- T -space.*

Proof. (1) Let B be any space. It is sufficient to show that $[A, \Omega B] \subset DG(A, \Omega B)$. Since X is a co- T^r -space for a map $r : X \rightarrow A$, we have that $DG(X, r, A; \Omega B) = [X; \Omega B]$. Thus we know, from Proposition 3.1(5), that $[A, \Omega B] = i^\# [X, \Omega B] = i^\# DG(X, r, A; \Omega B) \subset DG(A, ri, A; \Omega B) = DG(A, 1, A; \Omega B) = DG(A, \Omega B)$. (2) For any space B , we can obtain, from Proposition 3.4 and Proposition 3.1(1), that $[X, \Omega B] = DG(X, r, A; \Omega B) \subset DG(X, ir, X; \Omega B) = DG(X, 1, X; \Omega B) = DG(X, \Omega B)$. $\square$

Given maps $g : X \rightarrow A$, $g' : X' \rightarrow A'$, let $(s, r) : g' \rightarrow g$ be a map from g' to g , that is, the following diagram is commutative;

$$\begin{array}{ccc} X' & \xrightarrow{g'} & A' \\ r \downarrow & & s \downarrow \\ X & \xrightarrow{g} & A. \end{array}$$

It is a well known fact that $Y \xrightarrow{\iota} cY \rightarrow \Sigma Y$ is a cofibration, where $\iota(y) = [y, 1]$. Let $i_r : X \rightarrow C_r$ be the cofibration induced by $r : X' \rightarrow X$ from $\iota_{X'} : X' \rightarrow cX'$. Let $i_s : A \rightarrow C_s$ be the cofibration induced by $s : A' \rightarrow A$ from $\iota_{A'} : A' \rightarrow cA'$. Then there is a map $\bar{g} : C_t \rightarrow C_s$ such that the following diagram is commutative

$$\begin{array}{ccc} X & \xrightarrow{g} & A \\ i_r \downarrow & & i_s \downarrow \\ C_r & \xrightarrow{\bar{g}} & C_s, \end{array}$$

where $C_t = cX' \amalg X/[x', 1] \sim t(x')$, and $C_s = cA' \amalg A/[a', 1] \sim s(a')$, $\bar{g} : C_t \rightarrow C_s$ is given by $\bar{g}([x', t]) = [g'(x'), t]$ if $[x', t] \in cX'$ and $\bar{g}(x) = g(x)$ if $x \in X$, $i_r(x) = x$, $i_s(a) = a$.

DEFINITION 3.6. *Let X be a co- T^g -space with co- T^g -structure $\theta : X \rightarrow \Omega \Sigma X \vee A$. Then a map $(s, r) : g' \rightarrow g$ is called a co- T^g -primitive*

with respect to $\theta : X \rightarrow \Omega\Sigma X \vee A$ if there is a coassociate map $\theta' : X' \rightarrow \Omega\Sigma X' \vee A'$ of e' -cocyclic map g' such that the following diagram is homotopy commutative;

$$\begin{array}{ccc} X' & \xrightarrow{\theta'} & \Omega\Sigma X' \vee A' \\ r \downarrow & & \Omega\Sigma r \vee s \downarrow \\ X & \xrightarrow{\theta} & \Omega\Sigma X \vee A. \end{array}$$

The following lemmas are standard.

LEMMA 3.7. *Let $f : X \rightarrow B$ be a map. Then there is a map $h : C_r \rightarrow B$ such that $hi_r = f$ if and only if $fr \sim *$.*

LEMMA 3.8. [17] *Let $g_t : C_r \rightarrow B_t (t = 1, 2)$ and $g : C_r \rightarrow B_1 \vee B_2$ a map such that $p_t j g i_k \sim g_t i_r (t = 1, 2)$, where $j : B_1 \vee B_2 \rightarrow B_1 \times B_2$ is the inclusion and $p_t : B_1 \times B_2 \rightarrow B_t, t = 1, 2$ are projections. Then there is a map $h : C_r \rightarrow B_1 \vee B_2$ such that $g i_r = h i_r$ and $p_t j' h \sim g_t (t = 1, 2)$, where $j' : B_1 \vee B_2 \rightarrow B_1 \times B_2$ is the inclusion.*

THEOREM 3.9. *If X is a $co-T^g$ -space with $co-T^g$ -structure $\theta : X \rightarrow \Omega\Sigma X \vee A$ and $(s, r) : g' \rightarrow g$ is a $co-T^g$ -primitive with respect to θ , then there exists a $co-T^g$ -structure $\bar{\theta} : C_r \rightarrow \Omega\Sigma C_r \vee C_s$ on C_r satisfying commutative diagram*

$$\begin{array}{ccc} C_r & \xrightarrow{\bar{\theta}} & \Omega\Sigma C_r \vee C_s \\ i_r \uparrow & & \Omega\Sigma i_r \vee i_s \uparrow \\ X & \xrightarrow{\theta} & \Omega\Sigma X \vee A. \end{array}$$

Proof. Since $(s, r) : g' \rightarrow g$ is a $co-T^g$ -primitive with respect to θ , then there is a map $\theta' : X' \rightarrow \Omega\Sigma X' \vee A'$ satisfying commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\theta'} & \Omega\Sigma X' \vee A' \\ r \downarrow & & \Omega\Sigma r \vee s \downarrow \\ X & \xrightarrow{\theta} & \Omega\Sigma X \vee A. \end{array}$$

Then we have that $(\Omega\Sigma i_r \vee i_s)\theta r \sim (\Omega\Sigma i_r \vee i_s)(\Omega\Sigma r \vee s)\theta' \sim (\Omega\Sigma(i_r \circ r) \vee i_s \circ s)\theta \sim *$. Thus we know, from Lemma 3.7, that there is a map $\tilde{\theta} : C_r \rightarrow \Omega\Sigma C_r \vee C_s$ such that $\tilde{\theta} i_r = (\Omega\Sigma i_r \vee i_s)\theta$. Then $p_1 j \tilde{\theta} i_r = p_1 j (\Omega\Sigma i_r \vee i_s)\theta \sim p_1 (\Omega\Sigma i_r \times i_s)(e' \times g)\Delta \sim e'_{C_r} \circ i_r$ and $p_2 j \tilde{\theta} i_r \sim p_2 (\Omega\Sigma i_r \times i_s)(e' \times g)\Delta \sim i_s \circ g = \bar{g} \circ i_r$. Thus we have, from Lemma 3.8, that there is a map $\bar{\theta} : C_r \rightarrow \Omega\Sigma C_r \vee C_s$ such that $\bar{\theta} i_r = \tilde{\theta} i_r = (\Omega\Sigma i_r \vee i_s)\theta$

and $p_1 j \bar{\theta} \sim e'$, $p_2 j \bar{\theta} \sim \bar{g}$, where $j : \Omega \Sigma C_r \vee C_s \rightarrow \Omega \Sigma C_r \times C_s$ is the inclusion. $\square$

Taking $g = 1_X$, $g' = 1_{X'}$ and $s = r$, we can get the following corollary.

COROLLARY 3.10. *Let X and X' be co- T -spaces with co- T^1 -structures $\theta : X \rightarrow \Omega \Sigma X \vee X$ and $\theta' : X' \rightarrow \Omega \Sigma X' \vee X'$ respectively. If $r : X' \rightarrow X$ is a map satisfying $(\Omega \Sigma r \vee r)\theta' \sim \theta r : X' \rightarrow \Omega \Sigma X \vee X$, then there is a co- T^1 -structure $\bar{\theta} : C_r \rightarrow \Omega \Sigma C_r \vee C_r$ on C_r such that $(\Omega \Sigma i_r \vee i_r)\theta \sim \bar{\theta} i_r : X \rightarrow \Omega \Sigma C_r \vee C_r$.*

In 1959, Eckmann and Hilton [2] introduced a dual concept of Postnikov system as follows; A *homology decomposition* of X consists of a sequence of spaces and maps $\{X_n, q_n, i_n\}$ satisfying (1) $q_n : X_n \rightarrow X$ induces an isomorphism $(q_n)_* : H_i(X_n) \rightarrow H_i(X)$ for $i \leq n$. (2) $i_n : X_n \rightarrow X_{n+1}$ is a cofibration with cofiber $M(H_{n+1}(X), n)$ (a Moore space of type $(H_{n+1}(X), n)$). (3) $q_n \sim q_{n+1} \circ i_n$. It is known by [6] that if X be a 1-connected space having the homotopy type of CW complex, then there is a homology decomposition $\{X_n, q_n, i_n\}$ of X such that $i_n : X_n \rightarrow X_{n+1}$ is the principal cofibration induced from $\iota : M(H_{n+1}(X), n) \rightarrow cM(H_{n+1}(X), n)$ by a map $r : M(H_{n+1}(X), n) \rightarrow X_n$ which is called the dual Postnikov invariants.

From Theorem 3.9, we have the following corollary.

COROLLARY 3.11. *Let X and A be spaces having the homotopy type of 1-connected countable CW-complexes, and $\{X_n, q_n, i_n\}$ and $\{A_n, q'_n, i'_n\}$ be homology decompositions for X and A respectively. If X is a co- T^g -space with co- T^g -structure $\theta : X \rightarrow \Omega \Sigma X \vee A$ and for each $n \geq 2$, the pair of dual invariants $(r_A^n, r_X^n) : \tilde{g}_* \rightarrow g_n$ are co- T^{g_n} -primitive with respect to $\theta_n : X_n \rightarrow \Omega \Sigma X_n \vee A_n$, where $\tilde{g}_* : M(H_{n+1}(X), n) \rightarrow M(H_{n+1}(A), n)$ and g_n are induced maps from $g : X \rightarrow A$, then there exists a co- $T^{g_{n+1}}$ -structure on X_{n+1} such that $(i'_{n+1}, i_{n+1}) : g_n \rightarrow g_{n+1}$ is a co- $T^{g_{n+1}}$ -primitive with respect to $\theta_n : X_n \rightarrow X_n \vee A_n$.*

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