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FREE ACTIONS ON THE 3–DIMENSIONAL NILMANIFOLD

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Abstract. We study free actions of finite groups on the 3-dimensional nilmanifold and classify all such group actions, up to topological conjugacy. This work generalize Theorem 3.10 of [1].

1. Introduction

The general question of classifying finite group actions on a closed 3-manifold is very hard. However, Free actions of finite, cyclic and abelian groups on the 3-torus were studied in [4], [5] and [6], respectively. It is known ([3; Proposition 6.1.]) that there are 15 classes of distinct closed 3-dimensional manifolds M with a Nil-geometry up to Seifert local invariant. It is interesting that if a finite group acts freely on the 3-dimensional nilmanifold with the first homology $\mathbb{Z}^2$, then it is cyclic [2]. Free actions of finite abelian groups on the 3-dimensional nilmanifold with the first homology $\mathbb{Z}^2 \oplus \mathbb{Z}_p$ were classified in [1].

Let $\mathcal{H}$ be the 3–dimensional Heisenberg group; i.e. $\mathcal{H}$ consists of all 3×3 real upper triangular matrices with diagonal entries 1. That is,

$$\mathcal{H} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \ : \ x, y, z \in \mathbb{R} \right\}.$$

Thus $\mathcal{H}$ is a simply connected, 2-step nilpotent Lie group.

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For each integer $p > 0$, let

$$\Gamma_p = \left\{ \begin{bmatrix} 1 & l & \frac{n}{p} \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix} \middle| \ l, m, n \in \mathbb{Z} \right\}.$$

Then Γ_1 is the discrete subgroup of $\mathcal{H}$ consisting of all integral matrices and Γ_p is a lattice of $\mathcal{H}$ containing Γ_1 with index p. Clearly

$$\mathrm{H}_1(\mathcal{H}/\Gamma_p;\, \mathbb{Z}) = \Gamma_p/[\Gamma_p, \Gamma_p] = \mathbb{Z}^2 \oplus \mathbb{Z}_p.$$

Note that these ${\Gamma_p}'$s produce infinitely many distinct nilmanifolds $\mathcal{N}_p = \mathcal{H}/\Gamma_p$ covered by $\mathcal{N}_1$. Free actions of finite groups on the 3-dimensional nilmanifold which yield an orbit manifold homeomorphic to $\mathcal{H}/\pi$ were classified in [7], where $\pi = \langle t_1, t_2, t_3, \mid [t_2, t_1] = t_3^n,\ \ [t_3, t_1] = [t_3, t_2] = 1\rangle$.

In this paper, we shall find all possible finite groups acting freely on each $\mathcal{N}_p$ by utilizing the method used in [1] and classify all such group actions, up to topological conjugacy. We shall use all notations and most of the Introduction, Section 2 and Section 3 of [1]. This work generalize Theorem 3.10 of [1].

Let $\pi_i = \langle t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^K,\ \ [t_3, \alpha] = [t_3, t_1] = [t_3, t_2] = 1,$
$\alpha t_1 \alpha^{-1} = t_1 t_2,\ \alpha t_2 \alpha^{-1} = t_1^{-1},\ \alpha^6 = t_3^j \,\rangle,$

$$\alpha = \left(\begin{bmatrix} 1 & 0 & -\frac{j}{6K} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \right) \right),$$

where $1 \leq i \leq 4$, $K = 6n$ for the cases of π_1 and π_3, $K = 6n - 2$ for the case of π_2 and $K = 6n - 4$ for the case of π_4; $j = 1$ for the cases of π_1 and π_2, and $j = 5$ otherwise, be an almost Bieberbach group and N be a normal nilpotent subgroup of π_i with $G = \pi_i/N$ finite. For the almost Bieberbach group π_i, we find all normal nilpotent subgroups N of π_i, and classify (N, π_i) up to affine conjugacy.

2. Free actions of finite groups on the 3-dimensional nilmanifold

In this section, we shall find all possible finite groups acting freely (up to topological conjugacy) on the 3-dimensional nilmanifold $\mathcal{N}_p$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\pi_i$. This was done by the program MATHEMATICA[8] and hand-checked.

Lemma 1. *Let N be a normal nilpotent subgroup of an almost Bieberbach group $\pi_i (i = 1,2,3,4)$ and isomorphic to Γ_p. Then N can be represented by one of the following sets of generators*

$$N_1 = \langle t_1^{d_1} t_2^m,\ t_2^{d_2},\ t_3^{\frac{Kd_1d_2}{p}} \rangle, \qquad N_2 = \langle t_1^{d_1} t_2^m,\ t_2^{d_2} t_3^{\frac{Kd_1d_2}{2p}},\ t_3^{\frac{Kd_1d_2}{p}} \rangle,$$

$$N_3 = \langle t_1^{d_1} t_2^m t_3^{\frac{Kd_1d_2}{2p}},\ t_2^{d_2},\ t_3^{\frac{Kd_1d_2}{p}} \rangle, \qquad N_4 = \langle t_1^{d_1} t_2^m t_3^{\frac{Kd_1d_2}{2p}},\ t_2^{d_2} t_3^{\frac{Kd_1d_2}{2p}},\ t_3^{\frac{Kd_1d_2}{p}} \rangle,$$

where $\frac{d_1}{d_2} + \frac{m(m-d_1)}{d_1d_2} \in \mathbb{Z}$ *and* d_1 *is a common divisor of* m *and* d_2.

Proof. Let N be a normal nilpotent subgroup of $\pi_i (i = 1,2,3,4)$ and isomorphic to Γ_p. Then by Proposition 3.1 in [1],

$$N = \langle\, t_1^{d_1} t_2^m t_3^\ell,\ t_2^{d_2} t_3^r,\ t_3^{\frac{Kd_1d_2}{p}} \rangle, \quad \left(0 \le m < d_2,\ 0 \le \ell, r < \frac{Kd_1d_2}{p}\right),$$

where $K = 6n$ for $i = 1,3$, $K = 6n-2$ for $i = 2$ and $K = 6n-4$ for $i = 4$. Since N is a normal nilpotent subgroup of π_i, the following two relations

$$\alpha(t_1^{d_1} t_2^m t_3^\ell)\alpha^{-1} = (t_1t_2)^{d_1}(t_1^{-1})^m t_3^\ell = (t_1^{d_1} t_2^m t_3^\ell)^{\frac{d_1-m}{d_1}} (t_2^{d_2} t_3^r)^x (t_3^{\frac{Kd_1d_2}{p}})^y \in N,$$

$$\alpha(t_2^{d_2} t_3^r)\alpha^{-1} = t_1^{-d_2} t_3^r = (t_1^{d_1} t_2^m t_3^\ell)^{-\frac{d_2}{d_1}} (t_2^{d_2} t_3^r)^{\frac{m}{d_1}} (t_3^{\frac{Kd_1d_2}{p}})^z \in N$$

show that

$$x = \frac{d_1}{d_2} + \frac{m(m-d_1)}{d_1d_2} \in \mathbb{Z}, \quad \frac{m}{d_1} \in \mathbb{Z}, \quad \frac{d_2}{d_1} \in \mathbb{Z}.$$

Thus d_1 is a common divisor of m and d_2.

Let $\beta = \alpha^3$. Then the following two relations

$$\beta(t_1^{d_1} t_2^m t_3^\ell)\beta^{-1} = (t_1^{d_1} t_2^m t_3^\ell)^{-1} (t_3^{\frac{Kd_1d_2}{p}})^x \in N,$$

$$\beta(t_2^{d_2} t_3^r)\beta^{-1} = (t_2^{d_2} t_3^r)^{-1} (t_3^{\frac{Kd_1d_2}{p}})^y \in N$$

show that

$$x = \frac{2pl}{Kd_1d_2} - \frac{pm}{d_2} + \frac{p(m-d_1)}{d_1d_2}, \quad y = \frac{2pr}{Kd_1d_2} + \frac{p}{d_1}$$

must be integers. Therefore we can get $\frac{2pl}{Kd_1d_2} \in \mathbb{Z}$ and $\frac{2pr}{Kd_1d_2} \in \mathbb{Z}$. Since $0 \le l, r < \frac{Kd_1d_2}{p}$, we have $l = 0$ or $\frac{Kd_1d_2}{2p}$ and $r = 0$ or $\frac{Kd_1d_2}{2p}$. Therefore we have proved the lemma. $\square$

THEOREM 2. *Let N^m and $N^{m'}$ be normal nilpotent subgroups of π_i whose sets of generators are*

$$N^m = \langle\, {t_1}^{d_1} t_2^m t_3^\ell,\ t_2^{d_2} t_3^r,\ t_3^{\frac{Kd_1d_2}{p}}\,\rangle,$$

$$N^{m'} = \langle\, {t_1}^{d_1} t_2^{m'} t_3^{\ell'},\ t_2^{d_2} t_3^{r'},\ t_3^{\frac{Kd_1d_2}{p}}\,\rangle.$$

If $m \neq m'$, then N^m is not affinely conjugate to $N^{m'}$.

Proof. By applying the method used in Theorem 3.3 of [1], we can find the normalizer $N_{\mathrm{Aff}(\mathcal{H})}(\pi_i)$:

$$\mu(x,y,z,u,v) = \left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix},\ \left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\right),$$

where $x \in \mathbb{Z},\ \ y \in \mathbb{Z},\ \ z \in \mathbb{R}$ and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{Z}_6 \rtimes \mathbb{Z}_2 = \left\langle \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle.$$

Note that $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathrm{Aut}(\mathcal{H})$ can be evaluated respectively by the elements of $\mathbb{Z}_6 \rtimes \mathbb{Z}_2$. More precisely, the values of $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathrm{Aut}(\mathcal{H})$ are

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix},\ \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix},\ \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix},\ \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}.$$

For example, we can find

$$\mu(x,y,z,1,\frac{1}{2}) = \left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix},\ \left(\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}\right)\right) \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_i).$$

Assume that N^m is affinely conjugate to $N^{m'}$. Then there exists

$$\mu = \left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix},\ \left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\right) \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_i)$$

satisfying either

$$(*) \qquad \mu({t_1}^{d_1} {t_2}^m t_3^\ell)\mu^{-1} = t_1^{d_1} {t_2}^{m'} t_3^{\ell'},\quad \mu(t_2^{d_2} t_3^r)\mu^{-1} = t_2^{d_2} t_3^{r'},$$

or

$$(**) \qquad \mu({t_1}^{d_1}{t_2}^{m}t_3^{\ell})\mu^{-1} = t_2^{d_2}{t_3}^{r'}, \quad \mu(t_2^{d_2}t_3^{r})\mu^{-1} = t_1^{d_1}{t_2}^{m'}t_3^{\ell'}.$$

From $(*)$, we obtain the following relations:

$$bd_2 = 0, \quad dd_2 = d_2, \quad ad_1 + bm = d_1, \quad cd_1 + dm = m'.$$

Thus we have

$$b = 0, \quad d = 1, \quad a = 1, \quad cd_1 = m' - m.$$

Since

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} \in \mathbb{Z}_6 \rtimes \mathbb{Z}_2,$$

we have $c = 0$ and $m = m'$, which is a contradiction. However in $(**)$, we obtain the following relations:

$$bd_2 = d_1, \quad dd_2 = m', \quad ad_1 + bm = 0, \quad cd_1 + dm = d_2.$$

The relation $dd_2 = m' < d_2$ induces $m' = 0$ and $d = 0$. Since

$$cd_1 = cbd_2 = d_2, \quad bc = 1, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \in \mathbb{Z}_6 \rtimes \mathbb{Z}_2,$$

we have $a = 0$ and so $m = 0$, which is a contradiction. Therefore we complete the proof. □

In the following theorem, we show when affine conjugacy occurs among 4 types of normal nilpotent subgroups $N_j (j = 1, 2, 3, 4)$.

Theorem 3. *Let $N_j (j = 1, 2, 3, 4)$ be a normal nilpotent subgroup of $\pi_i (i = 1, 2, 3, 4)$ and isomorphic to Γ_p. Then we have the following:*

(1) $N_2 \sim N_3$ if and only if $m = 0,\ d_1 = d_2$.

(2) $N_1 \nsim N_2, \quad N_1 \nsim N_4, \quad N_3 \nsim N_4$.

Proof. (1) Suppose that N_2 is affinely conjugate to N_3. Then there exists

$$\mu = \left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \right) \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_i)$$

satisfying either

$$(*)\qquad \mu({t_1}^{d_1}{t_2}^{m})\mu^{-1}=t_1^{d_1}{t_2}^{m}{t_3}^{\frac{Kd_1d_2}{2p}},\quad \mu(t_2^{d_2}{t_3}^{\frac{Kd_1d_2}{2p}})\mu^{-1}=t_2^{d_2},$$

or

$$(**)\qquad \mu({t_1}^{d_1}{t_2}^{m})\mu^{-1}=t_2^{d_2},\quad \mu(t_2^{d_2}{t_3}^{\frac{Kd_1d_2}{2p}})\mu^{-1}=t_1^{d_1}{t_2}^{m}{t_3}^{-\frac{Kd_1d_2}{2p}}.$$

From $(*)$, we obtain the following relations:

$$bd_2=0,\quad dd_2=d_2,\quad ad_1+bm=d_1,\quad cd_1+dm=m.$$

Thus we have $\begin{bmatrix} a & b\\ c & d\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ and $x=\frac{d_1}{2p}$. Note that d_1 is a divisor of p. Since $\mu\in N_{\mathrm{Aff}(\mathcal{H})}(\pi_i)$, we have $x=\frac{d_1}{2p}\in\mathbb{Z}$, which is a contradiction. However in $(**)$, we obtain the following relations:

$$bd_2=d_1,\quad dd_2=m,\quad ad_1+bm=0,\quad cd_1+dm=d_2.$$

The relation $dd_2=m<d_2$ induces $m=0$ and $d=0$. Thus we have $a=0$ and $b=c=1$. Therefore we have $d_1=d_2$ and $m=0$.

Conversely, suppose that $m=0$ and $d_1=d_2$. Then $N_2\sim N_3$ by using

$$\left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\left(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)\right)\in N_{\mathrm{Aff}(\mathcal{H})}(\pi_i).$$

(2) Suppose that N_1 is affinely conjugate to N_2. Then there exists $\mu\in N_{\mathrm{Aff}(\mathcal{H})}(\pi_i)$ satisfying either

$$(*)\qquad \mu({t_1}^{d_1}{t_2}^{m})\mu^{-1}=t_1^{d_1}{t_2}^{m},\quad \mu(t_2^{d_2})\mu^{-1}=t_2^{d_2}{t_3}^{\frac{Kd_1d_2}{2p}},$$

or

$$(**)\qquad \mu({t_1}^{d_1}{t_2}^{m})\mu^{-1}=t_2^{d_2}{t_3}^{\frac{Kd_1d_2}{2p}},\quad \mu(t_2^{d_2})\mu^{-1}=t_1^{d_1}{t_2}^{m}.$$

From $(*)$, we obtain that

$$\begin{bmatrix} a & b\\ c & d\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad x=-\frac{d_1}{2p},\quad y=-\frac{m}{2p}.$$

Since $\mu\in N_{\mathrm{Aff}(\mathcal{H})}(\pi_i)$, we have $x=-\frac{d_1}{2p}\in\mathbb{Z}$, which is a contradiction.

From $(**)$, we obtain the following relations:

$$bd_2=d_1,\quad dd_2=m,\quad ad_1+bm=0,\quad cd_1+dm=d_2.$$

The relation $dd_2 = m < d_2$ induces $d = 0$ and $m = 0$. Thus we have

$$a = 0, \quad b = c = 1, \quad x = -\frac{d_1}{2p}.$$

Since $\mu \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_i)$, we have $x = -\frac{d_1}{2p} \in \mathbb{Z}$, which is a contradiction. Therefore N_1 is not affinely conjugate to N_2.
The other cases can be done similarly. □

Note that π_i/N is abelian if and only if $N \supset [\pi_i, \pi_i] = \langle t_1,\ t_2,\ t_3^K\rangle$, where $K = 6n$ for $i = 1, 3$, $K = 6n - 2$ for $i = 2$ and $K = 6n - 4$ for $i = 4$. Thus we obtain the following result, which is the same as Theorem 3.10 of [1].

Corollary 4. *The following table gives a complete list of all free actions(up to topological conjugacy) of finite abelian groups G on $\mathcal{N}_p$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\pi_i$, $(i = 1, 2, 3, 4)$.*

Group G	AC classes of normal nilpotent subgroups	
$\mathbb{Z}_{\frac{6K}{p}}$	$\frac{K}{p} \in \mathbb{N}$	$N = \langle t_1,\ t_2,\ t_3^{\frac{K}{p}}\rangle$

References

1. D. Choi and J. K. Shin, *Free actions of finite abelian groups on 3-dimensional nilmanifolds*, J. Korean Math. Soc. **42(4)** (2005), 795–826.
2. H. Y. Chu and J. K. Shin, *Free actions of finite groups on the 3-dimensional nilmanifold*, Topology Appl. **144** (2004), 255–270.
3. K. Dekimpe, P. Igodt, S. Kim and K. B. Lee, *Affine structures for closed 3-dimensional manifolds with nil-geometry*, Quarterly J. Math. Oxford (2) **46** (1995), 141–167.
4. K. Y. Ha, J. H. Jo, S. W. Kim and J. B. Lee, *Classification of free actions of finite groups on the* 3-*torus*, Topology Appl. **121(3)** (2002), 469–507.
5. J. Hempel, *Free cyclic actions of* $S^1 \times S^1 \times S^1$, Proc. Amer. Math. Soc. **48(1)** (1975), 221–227.
6. K. B. Lee, J. K. Shin and Y. Shoji, *Free actions of finite abelian groups on the 3-Torus*, Topology Appl. **53** (1993), 153–175.
7. J. K. Shin, *Free actions of finite groups on the 3-dimensional nilmanifold for Type 1*, J. Chungcheong Math. Soc. **19(4)** (2006), 437–443.
8. S. Wolfram, *Mathematica*, Wolfram Research, 1993.

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