JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **20**, No. 3, September 2007

# SOME STRUCTURES ON A COMPLETE LATTICE

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Abstract. In this paper, we define $\bigwedge$-structure, $\bigvee$-structure to generalize some lattices, and study the conditions that a lattice which has $\bigwedge$-structure or $\bigvee$-structure to be continuous or algebraic.

## 1. Preliminaries

A set $P$ equipped with a partial order '$\leq$' is said to be a *poset*.
For any set $X$, the set of all finite subsets of $X$ is denoted by $FinX$.

Definition 1.1. Let $P$ be a poset and $D \subseteq P$. $D$ is said to be *directed* if $X$ has an upper bound in $D$ for every $X \in FinD$.

By the definition of a directed set every directed set is nonempty, because $\emptyset \in FinD$.
A lattice means a bounded lattice in this paper, i. e., a lattice which has 0 and $e$.
For terminology not introduced in this paper, we refer to [2, 4].

Lemma 1.2. *Let $P$ be a poset. Then $P$ has $\bigwedge S$ ($\bigvee S$, resp.) for every $S \subseteq P$ if and only if $P$ is a complete lattice with $\bigvee S = \bigwedge S^u$ ($\bigwedge S = \bigvee S^l$, resp.), where $S^u$ ($S^l$, resp.) is the set of all upper bounds (all lower bounds, resp.) of $S$.*

Further discussions of the fundamentals of a lattice can be found in [1, 3, 5].

Definition 1.3. Let $L$ be a complete lattice. Then we have the following:
(1) $L$ is called *meet-continuous* if it satisfies : $x \wedge (\bigvee D) = \bigvee\{x \wedge d \mid d \in D\}$ for all $x \in L$ and all directed subset $D$ of $L$.
(2) $L$ is called a *frame* if it satisfies : $x \wedge (\bigvee S) = \bigvee\{x \wedge s \mid s \in S\}$ for all $x \in L$ and all $S \subseteq L$.

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Received June 13, 2007.
2000 Mathematics Subject Classification: Primary 06A06, 06A11, 06A15.
Key words and phrases: adjunction, atomistic lattice, $\bigwedge$-structure, $\bigvee$-structure.

(3) For $x, y \in L$, $x$ is said to be *way below* $y$, denoted by $x \ll y$, if for any directed subset $D$ of $L$ with $y \leq \bigvee D$, there is $d \in D$ with $x \leq d$.

If $x \ll x$, then $x$ is said to be a *compact element of* $L$.

(4) $L$ is said to be *continuous* if $x = \bigvee \downarrow_w x$ for all $x \in L$, where $\downarrow_w x = \{y \in L | y \ll x\}$ .

(5) $L$ is said to be *algebraic* if $x = \bigvee(\downarrow x \cap K(L))$ for all $x \in L$, where $\downarrow x = \{y \in L \mid y \leq x\}$ and $K(L) = \{x \in L | x \ll x\}$.

DEFINITION 1.4. Let $P$ and $Q$ be posets. A map $f : P \to Q$ is said to be

(1) *monotone* if $x \leq y$ in $P$ implies $f(x) \leq f(y)$ in $Q$;
(2) *order-embedding* if $x \leq y$ in $P$ if and only if $f(x) \leq f(y)$ in $Q$;
(3) *order-isomorphism* if it is an order-embedding map $P$ onto $Q$.

If there exists an order-isomorphism from $P$ to $Q$, then we say that $P$ and $Q$ are *isomorphic* and write $P \cong Q$.

If $f : P \to Q$ is an order-embedding map, then $f$ is a monotone 1-1 map. But the converse need not be true. For $P = \{0, a, b, e\}$ with $0 < a < e$, $0 < b < e$ and $a \| b$ (i.e., $a$ and $b$ are non-comparable) and $Q = \{0, x, y, e\}$ with $0 < x < y < e$, a map $f : P \to Q$ defined by: $f(0) = 0,\ f(a) = x,\ f(b) = y,\ f(e) = e$, is monotone and 1-1, but not order-embedding, because $f(a) \leq f(b)$ but $a \| b$.

DEFINITION 1.5. Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be posets. We say that a pair $(f, g)$ of maps $f : P \to Q$ and $g : Q \to P$ is an *adjunction* (or *Galois connection*), denoted by $f \dashv g$, between $P$ and $Q$ if

(1) $f$ and $g$ are monotone,
(2) $f(p) \leq_Q q$ if and only if $p \leq_P g(q)$ for all $p \in P$, $q \in Q$.

If $(f, g)$ is an adjunction, then $f$ is called the *left adjoint* of $g$ and $g$ is the *right adjoint* of $f$.

DEFINITION 1.6. A map $c : P \to P$ is said to be a *closure operator* on a poset $P$ if $c$ is a monotone, idempotent self map with $1_P \leq c$.

EXAMPLE 1.7. (1) Let $P$ be a poset. Then $c : P \to P$ is a closure operator on $P$ if and only if $(c^\circ, i)$ is an adjunction between $P$ and $c(P)$, where $i : c(P) \to P$ is the inclusion map and $c^\circ : P \to c(P)$ is the corestriction of $c$.

(2) Let $X = \{1, 2, 3\}$ and let $P = \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set lattice of $X$. We define a map $c : P \to P$ by $c(x) = x \cup \{2\}$ for each $x \in P$. Then $c$ is a closure operator.

Let $L$ and $K$ be complete lattices. If $f : L \to K$ is a map, then we define two maps $f^l : K \to L$ and $f^r : K \to L$ by :

$f^l(y) = \bigwedge\{x \in L \mid y \leq f(x)\}$ and $f^r(y) = \bigvee\{x \in L \mid f(x) \leq y\}$, for each $y \in K$.

For any map $f : L \to K$ of complete lattices $L$ and $K$, $f^l$ ($f^r$, resp.) need not be the left adjoint (the right adjoint, resp.) of $f$.

An order-embedding map on complete lattices preserves neither arbitrary joins (including $e$) nor arbitrary meets (including 0) in general.

Example 1.8. (1) Let $\mathcal{U}$ be the usual topology on the set of all real numbers $\mathbb{R}$. If $i : \mathcal{U} \to \mathcal{P}(\mathbb{R})$ is the inclusion map, then $i$ is an order-embedding and preserves the top $\mathbb{R}$ and the bottom $\emptyset$.

If $S = \{(1 - 1/n, 1 + 1/n) \in \mathcal{U} \mid n \in \mathbb{N}\}$, then $\bigwedge_{\mathcal{U}} S = int(\bigcap S) = \emptyset$, where $int(A)$ is the interior of $A$, hence $i(\bigwedge_{\mathcal{U}} S) = i(\emptyset) = \emptyset$, but $\bigwedge_{\mathcal{P}(\mathbb{R})} i(S) = \bigcap\{(1 - 1/n, 1 + 1/n) \mid n \in \mathbb{N}\} = \{1\}$. Hence $i$ does not preserve arbitrary meets. If $g : \mathcal{P}(\mathbb{R}) \to \mathcal{U}$ is the map given by $g(X) = int(X)$ for all $X \in \mathcal{P}(\mathbb{R})$. Then $g$ preserves arbitrary meets, so $g$ has the left adjoint $i$. Hence $i$ preserves arbitrary joins.

The map $i^l : \mathcal{P}(\mathbb{R}) \to \mathcal{U}$ is not the left adjoint of $i$, because $ii^l(\{1\}) = \bigwedge\{G \in \mathcal{U} \mid \{1\} \subseteq G\} = \emptyset$ and $1_{\mathcal{P}}(\mathbb{R})(\{1\}) = \{1\}$, that is, $1_{\mathcal{P}(\mathbb{R})} \not\leq ii^l$.

(2) Let $V$ be the vector space $\mathbb{R}^2$ over the real numbers $\mathbb{R}$ and $SubV$ a complete lattice of all subspaces of $V$, and let $V_{\mathbf{x}}$ be the subspaces of $V$ generated by a vector $\mathbf{x}$. If $i : SubV \to \mathcal{P}(V)$ is the inclusion map, then $i$ is an order-embedding and preserves the top $V$, but it does not preserve the bottom, because $i(\{(0,0)\}) = \{(0,0)\} \neq \emptyset$.

If $\mathbf{i} = (1,0)$, $\mathbf{j} = (0,1)$, then $V_{\mathbf{i}} \vee_{SubV} V_{\mathbf{j}} = \mathbb{R}^2$ since $\mathbb{R}^2$ is the smallest subspace containing $V_{\mathbf{i}}$ and $V_{\mathbf{j}}$, hence $i(V_{\mathbf{i}}) \vee_{\mathcal{P}(V)} i(V_{\mathbf{j}}) = V_{\mathbf{i}} \cup V_{\mathbf{j}} \neq \mathbb{R}^2 = i(\mathbb{R}^2) = i(V_{\mathbf{i}} \vee_{SubV} V_{\mathbf{j}})$. Let $g : \mathcal{P}(V) \to SubV$ be the map given by $g(X) = [X]$ for all $X \in \mathcal{P}(V)$, where $[X]$ is the subspace of $V$ generated by the set $X$.

Then $i$ is the right adjoint of $g$, and hence $i$ preserves arbitrary meets.

The map $i^r : \mathcal{P}(V) \to SubV$ is not the right adjoint of $i$, because $ii^r(\{\mathbf{i}\})V_{\mathbf{i}} \not\leq \{\mathbf{i}\} = 1_{\mathcal{P}(V)}(\{\mathbf{i}\})$, that is, $ii^r \not\leq 1_{\mathcal{P}(V)}$.

We remark that the complete lattice $SubV$ of all subspaces of a vector space $V$ is not a distributive lattice, hence not a frame. In fact, if $\mathbf{i} = (1,0)$, $\mathbf{j} = (0,1)$ and $\mathbf{x} = (1,1)$ in the above example (2), then $V_{\mathbf{x}} \wedge (V_{\mathbf{i}} \vee V_{\mathbf{j}}) = V_{\mathbf{x}} \wedge \mathbb{R}^2 = V_{\mathbf{x}}$, and $(V_{\mathbf{x}} \wedge V_{\mathbf{i}}) \vee (V_{\mathbf{x}} \wedge V_{\mathbf{j}}) = \{(0,0)\} \vee \{(0,0)\} = \{(0,0)\}$. But $SubV$ is a meet-continuous lattice. So we can conclude that a meet-continuous lattice need not be distributive.

EXAMPLE 1.9. Let $P = \{0, a, e\}$ be a chain with the partial order : $0 < a < e$, and let $Q = \{0, x, y, e\}$ be a chain with the partial order : $0 < x < y < e$.

(1) If $f : P \to Q$ is the map defined by : $f(0) = 0,\ f(a) = x,\ f(e) = e$, then $f$ is order-embedding. If $g : Q \to P$ is the map defined by : $g(0) = 0,\ g(x) = a,\ g(y) = a,\ g(e) = e$, then $g$ is a left inverse of $f$, but $(fg)(y) = x < y = 1_Q(y)$, that is, $1_Q \not\leq fg$. It follows that $g$ is not the left adjoint of $f$.

(2) If $f : P \to Q$ is the map defined by : $f(0) = 0,\ f(a) = y,\ f(e) = e$, then $f$ is order-embedding. If $g : Q \to P$ is the map defined by : $g(0) = 0,\ g(x) = a,\ g(y) = a,\ g(e) = e$, then $g$ is a left inverse of $f$, but $1_Q(x) = x < y = (fg)(x)$, that is, $fg \not\leq 1_Q$. It follows that $g$ is not the right adjoint of $f$

Let $P$ and $Q$ be posets and $f : P \to Q$ a monotone map. If $D$ is a directed subset of $P$, then $f(D)$ is a directed subset of $Q$. Thus we have:

PROPOSITION 1.10. *Let $L$ and $K$ be complete lattices and $f : L \to K$ an order-embedding map which preserves directed joins.*
*If $f(x) \ll f(y)$ in $K$, then $x \ll y$ in $L$.*

*Proof.* Suppose that $f(x) \ll f(y)$ in $K$ and let $D$ be a directed subset of $L$ with $y \leq \bigvee D$. Since $f$ is monotone and preserves directed joins, $f(y) \leq f(\bigvee D) = \bigvee f(D)$ and $f(D)$ is a directed subset of $K$.

Since $f(x) \ll f(y)$, there is $d \in D$ with $f(x) \leq f(d)$, and it implies $x \leq d \in D$ because $f$ is order-embedding. Hence $x \ll y$. □

PROPOSITION 1.11. *Let $L$ and $K$ be complete lattices and $f : L \to K$ a map with $f^l \dashv f$ which preserves directed joins. If $x \ll y$ in $K$, then $f^l(x) \ll f^l(y)$ in $L$.*

*Proof.* Suppose that $x \ll y$ in $K$ and $D$ is a directed subset of $L$ with $f^l(y) \leq \bigvee D$. Since $f^l \dashv f$ and $f$ preserves directed joins, $y \leq f(\bigvee D) = \bigvee f(D)$ and $f(D)$ is a directed subset of $K$.

Since $x \ll y$, there is $d \in D$ with $x \leq f(d)$, and it implies $f^l(x) \leq d \in D$. Hence $f^l(x) \ll f^l(y)$. □

The converse of the above proposition is not true in general.

EXAMPLE 1.12. In Example 1.8-(2), the inclusion map $i : SubV \to \mathcal{P}(V)$ is an order-embedding and $i^l = g \dashv i$.

If $\mathcal{D}$ is a directed subset of $SubV$, then $\bigcup \mathcal{D} \in SubV$, because if $u, v \in \bigcup \mathcal{D}$, then there are $D_u$ and $D_v$ in $SubV$ with $u \in D_u$ and $v \in D_v$. Since $\mathcal{D}$ is directed, there is $D \in \mathcal{D}$ with $D_u, D_v \subseteq D$, hence $u, v \in D$, and $u + v \in D \subseteq \bigcup \mathcal{D}$. And it is clear that $\alpha v \in \bigcup \mathcal{D}$ for all $\alpha \in F$ and all $v \in \bigcup \mathcal{D}$.

That is, we have $\bigvee_{SubV} \mathcal{D} = [\bigcup \mathcal{D}] = \bigcup \mathcal{D} = \bigvee_{\mathcal{P}(V)} \mathcal{D}$. Hence $i$ preserves directed joins. It is clear that $V_{\mathbf{i}} \ll V$ in $SubV$. But $V_{\mathbf{i}} \not\ll V$ in $\mathcal{P}(V)$.

In fact, let $\mathcal{A} = \{\{\mathbf{x}\} \in \mathcal{P}(V) \mid \mathbf{x} \in V\}$ and let $\mathcal{F} = \{\vee F \mid F \in Fin\mathcal{A}\} \subseteq \mathcal{P}(V)$. Then $\mathcal{F}$ is a directed subset of $\mathcal{P}(V)$ with $V \leq \bigvee \mathcal{F}$, but $V_{\mathbf{i}} \not\leq \vee F = \cup F$ for all $F \in Fin\mathcal{A}$ since $\cup F$ is finite subset of $V$.

Hence we have that $i^l(V_{\mathbf{i}}) = V_{\mathbf{i}} \ll V = i^l(V)$ in $SubV$, but $i(V_{\mathbf{i}}) = V_{\mathbf{i}} \not\ll V = i(V)$ in $\mathcal{P}(V)$.

Let $L$ and $K$ be complete lattices and $f : L \to K$ an order-embedding map with $f^l \dashv f$. If $f^l_\circ : f(L) \to L$ is the restriction of $f^l$ to $f(L)$, then $f^l_\circ$ is an order-embedding since $f^l_\circ f = 1_L$.

Proposition 1.13. *Let $L$ and $K$ be complete lattices and $f : L \to K$ an order-embedding map which preserves directed joins with $f^l \dashv f$ . Then $y \ll f^\circ(x)$ in $f(L)$ if and only if $f^l_\circ(y) \ll x$ in $L$, where $f^\circ : L \to f(L)$ is the corestriction of $f$.*

*Proof.* Let $y \ll f^\circ(x)$ in $f(L)$. Then $f^l_\circ(y) = f^l(y) \ll x$.

Conversely, suppose that $f^l_\circ(y) \ll x$ in $L$ and $D$ is a directed subset of $f(L)$ with $f^\circ(x) \leq \bigvee D$. Then we have $x = 1_L(x) = f^l_\circ f^\circ(x) \leq f^l_\circ(\bigvee D) = \bigvee f^l_\circ(D)$. Since $f^l_\circ(y) \ll x$ and $f^l_\circ(D)$ is a directed subset of $L$, there is $d \in D$ with $f^l_\circ(y) \leq f^l_\circ(d)$. so $y \leq d \in D$ since $f^l_\circ$ is order-embedding. Hence $y \ll f^\circ(x)$. □

## 2. $\bigwedge$-structure and $\bigvee$-structure

Let $(P, \leq)$ be a poset and $x, y \in P$. We say that $x$ is *covered by* $y$ (or $y$ *covers* $x$), denoted by $x \prec y$ or $y \succ x$, if $x < y$ and $x \leq z < y$ implies $z = x$.

Let $L$ be a lattice and $a \in L$. $a$ is called an *atom* if $0 \prec a$. We denote the set of all atoms in $L$ by $A(L)$.

A lattice $L$ with atoms is said to be *atomistic* if $x = \bigvee(\downarrow x \cap A(L))$ for all $x \in L$.

The power set lattice $\mathcal{P}(L)$ of a complete lattice $L$ is a frame; hence a meet-continuous lattice, which is atomistic with $A(\mathcal{P}(L)) = \{\{x\} \mid x \in L\}$. And the map $\downarrow: L \to \mathcal{P}(L)$ $(x \mapsto \downarrow x)$ is 1-1 and has the left adjoint $\bigvee : \mathcal{P}(L) \to L$ $(S \mapsto \bigvee S)$, that is, it is 1-1 meet-preserving map. Hence we can consider a map (or 1-1 meet-preserving map) from $L$ to an atomistic meet-continuous lattice $H$.

Lemma 2.1. *Let $H$ be an atomistic meet-continuous lattice. If $a \in A(H)$ with $a \le \bigvee D$ for any directed subset $D$ of $H$, then there is $d \in D$ with $a \le d$.*

*Proof.* Let $D$ be a directed subset of an atomistic meet-continuous lattice $H$ and $a$ an atom with $a \le \bigvee D$. Then $a = a \wedge (\bigvee D) = \bigvee\{a \wedge d \mid d \in D\}$. Since $0 \le a \wedge d \le a$ for all $d \in D$ and $a$ is an atom, $a \wedge d = a$ or $a \wedge d = 0$ for each $d \in D$.

If $a \wedge d = 0$ for all $d \in D$, then $a = \bigvee\{a \wedge d \mid d \in D\} = 0$. It is a contradiction to $a \ne 0$. Hence there is $d \in D$ with $a \wedge d = a$, that is, $a \le d$. □

By the above lemma, every atom in an atomistic meet-continuous lattice $H$ is compact, that is, $A(H) \subseteq K(H)$.

Let $L$ be a complete lattice and $H$ an atomistic complete lattice. If $f : L \to H$ is a map, then we denote $E_f(L) = \{f^l(a) \in L \mid a \in A(H)\}$.

Lemma 2.2. *Let $L$ be a complete lattice and $H$ an atomistic complete lattice.*

*If $f : L \to H$ a map with $f^l \dashv f$, then for any $x \in L$ we have the following :*

(1) $\downarrow x \cap E_f(L) = f^l(\downarrow f(x) \cap A(H))$,
(2) $f(x) = \bigvee(\downarrow f(x) \cap A(H)) \le \bigvee f(\downarrow x \cap E_f(L))$.

*Proof.* (1) Let $u \in \downarrow x \cap E_f(L)$. Then there is $a \in A(H)$ with $f^l(a) = u \le x$.

Since $f^l \dashv f$, $a \le f(x)$. Hence $a \in \downarrow f(x) \cap A(H)$ and $u = f^l(a) \in f^l(\downarrow f(x) \cap A(H))$.

Conversely, let $u \in f^l(\downarrow f(x) \cap A(H))$. Then there is $a \in A(H)$ with $a \le f(x)$ and $u = f^l(a)$. Since $f^l \dashv f$, $u = f^l(a) \le x$. Hence $u \in \downarrow x \cap E_f(L)$.

(2) It is clear that $f(x) = \bigvee(\downarrow f(x) \cap A(H))$ since $H$ is an atomistic lattice.

We need to show that $\bigvee(\downarrow f(x) \cap A(H)) \le \bigvee f(\downarrow x \cap E_f(L))$.

Since $1_H \le ff^l$ and $\downarrow x \cap E_f(L) = f^l(\downarrow f(x) \cap A(H))$ by (1), we have $\bigvee(\downarrow f(x) \cap A(H)) \le \bigvee ff^l(\downarrow f(x) \cap A(H)) = \bigvee f(\downarrow x \cap E_f(L))$. □

Proposition 2.3. *Let $L$ be a complete lattice, $H$ an atomistic meet-continuous lattice and $f : L \to H$ a map which preserves directed joins with $f^l \dashv f$.*

*If $x \le \vee_L F \le y$ for some $F \in Fin(\downarrow y \cap E_f(L))$, then $x \ll y$ in $L$.*

*Proof.* Suppose that $F \in Fin(\downarrow y \cap E_f(L))$ with $x \leq \vee_L F \leq y$ and $D$ is a directed subset of $L$ with $y \leq \bigvee D$. For each $u \in F$, there is $a_u \in A(H)$ with $f^l(a_u) = u$. Since $f$ is monotone which preserves directed joins with $f^l \dashv f$, $a_u \leq f(u) \leq f(y) \leq f(\bigvee_L D) = \bigvee_H f(D)$. Since $f(D)$ is directed, there is $d \in D$ with $a_u \leq f(d)$ .

Choose one element $d_u$ in $\{d \in D \mid a_u \leq f(d)\}$ and let $S = \{d_u \in D \mid u \in F\}$ for each $u \in F$. Since $a_u \leq f(d_u)$ for all $d_u \in S$ and $f^l \dashv f$, $u = f^l(a_u) \leq d_u$ for each $u \in F$. so $x \leq \vee F = \vee S$. Since $S$ is a finite subset of $D$, there is $d \in D$ with $d_u \leq d$ for all $d_u \in S$, so $\vee S \leq d$ and $x \leq d \in D$. Hence $x \ll y$. □

Proposition 2.4. *Let $L$ be a complete lattice, $H$ an atomistic meet-continuous lattice and $f : L \to H$ a map with $f^l \dashv f$. Then $f$ preserves directed joins if and only if every element of $E_f(L)$ is compact.*

*Proof.* Suppose that $f$ preserves directed joins and $u \in E_f(L)$. Let $D$ be a directed subset of $L$ with $u \leq \bigvee D$. Then there is $a \in A(H)$ with $f^l(a) = u$. Since $f^l \dashv f$ and $f$ preserves directed joins, $a \leq f(u) \leq f(\bigvee_L D) = \bigvee_H f(D)$. There is $d \in D$ with $a \leq f(d)$. Hence $u = f^l(a) \leq f^l(f(d)) \leq d$, so $u \ll u$.

Conversely, suppose that $u$ is a compact element for every $u \in E_f(L)$ and $D$ is a directed subset of $L$. Since $f(\bigvee_L D) \geq \bigvee_H f(D)$, it remain to show that $f(\bigvee_L D) \leq \bigvee_H f(D)$.

Let $\alpha = \bigvee_L D$ and $a \in \downarrow f(\alpha) \cap A(H)$. Then $a \leq f(\alpha)$ implies $f^l(a) \leq \alpha = \bigvee_L D$. Since $f^l(a)$ is a compact element in $L$ and $D$ is directed, there is $d \in D$ with $f^l(a) \leq d$, hence $a \leq f(d) \leq \bigvee_H f(D)$. That is, $a \leq \bigvee_H f(D)$ for every $a \in \downarrow f(\alpha) \cap A(H)$, so $f(\bigvee_L D) = f(\alpha) = \bigvee_H(\downarrow f(\alpha) \cap A(H)) \leq \bigvee_H f(D)$. □

Definition 2.5. Let $L$ and $H$ be complete lattices. Then

(1) $L$ is said to have a $(\bigwedge, f)$-*structure* in $H$ if there is an 1-1 map $f : L \to H$ which preserves arbitrary meets.
(2) $L$ is said to have a $(\bigvee, f)$-*structure* in $H$ if there is an 1-1 map $f : L \to H$ which preserves arbitrary joins.

If $L$ has a $(\bigwedge, f)$-structure ($(\bigvee, f)$-structure, resp.) in $H$, then $f$ is an order-embedding map and preserves the top element (the bottom element, resp.); $f(e_L) = f(\wedge_L \emptyset) = \wedge_H f(\emptyset) = \wedge_H \emptyset = e_H$. But $f$ need not preserve arbitrary joins (meets, resp.) as Example 1.8.

Definition 2.6. Let $L$ and $H$ be complete lattices with $L \subseteq H$ and $i : L \to H$ the inclusion map.

(1) $L$ is said to have a $\bigwedge$-*structure* in $H$ if $i$ preserves arbitrary meets,

(2) $L$ is said to have a $\bigvee$-*structure* in $H$ if $i$ preserves arbitrary joins.

Let $L$ have a $\bigwedge$-structure ($\bigvee$-structure, resp.) in $H$. Then $L$ has a $(\bigwedge, i)$-structure $((\bigvee, i)$-structure, resp.) in $H$ since $i$ is 1-1, and $\bigwedge_L S = \bigwedge_H S$ ($\bigvee_L S = \bigvee_H S$, resp) for every $S \subseteq L$, because $\bigwedge_L S = i(\bigwedge_L S) = \bigwedge_H i(S) = \bigwedge_H S$.

Lemma 2.7. *If $L$ has a $\bigwedge$-structure ($\bigvee$-structure, resp.) in a complete lattice $H$, then for any $S \subseteq L$, $\bigvee_L S \geq \bigvee_H S$ ($\bigwedge_L S \leq \bigwedge_H S$, resp.) in $H$.*

*Proof.* Let $i : L \to H$ be the inclusion map. Then $i$ is monotone, hence, $\bigvee_L S = i(\bigvee_L S) \geq \bigvee_H i(S) = \bigvee_H S$ for any $S \subseteq L$.

The proof of $\bigwedge_L S \leq_H \bigwedge_H S$ for a $\bigvee$-structure is the dual of this proof. □

Let $L$ have a $\bigwedge$-structure in $H$. Then the inclusion map $i : L \to H$ preserves arbitrary meet and $i^l \dashv i$. Since $x \leq x$ in for all $x \in L$, $i^l(x) = \bigwedge\{u \in L \mid x \leq i(u) = u\} = x$ for all $x \in L$, that is, $i^l_\circ = 1_L$, where $i^l_\circ : L \to L$ is the restriction of $i^l$ to $L$.

We denote $\hat{a} = i^l(a)$ for each $a \in A(H)$, that is, $\hat{a} = \bigwedge\{u \in L \mid a \leq u\}$, and $E(L) = E_i(L) = \{\hat{a} \in L \mid a \in A(H)\}$.

Proposition 2.8. *Let $L$ have a $\bigwedge$-structure in an atomistic meet-continuous lattice $H$. Then $\bigvee_H D \in L$ for every directed subset $D$ of $L$ if and only if $\hat{a}$ is compact for every $\hat{a} \in E(L)$.*

*Proof.* Let $D$ be a directed subset of $L$ and $\bigvee_H D \in L$. Since $D \subseteq L$, $i^l(d) = d$ for all $d \in D$, hence $i^l(D) = D$. Since $\bigvee_H D \in L$ and $i^l$ preserves arbitrary joins, $\bigvee_H D = i^l(\bigvee_H D) = \bigvee_L i^l(D) = \bigvee_L D$. Hence we have $i(\bigvee_L D) = \bigvee_L D = \bigvee_H D = \bigvee_L i(D)$, that is, $i$ preserves directed joins, and we have the equivalence of this proposition. □

Example 2.9. (1) The complete lattice $IdR$ of all ideals of a ring $R$ has a $\bigwedge$-structure in $\mathcal{P}(R)$ and for every directed subset $\mathcal{D}$ of $IdR$, $\bigvee_{\mathcal{P}(R)} \mathcal{D} = \cup\mathcal{D} \in IdR$. By Proposition 2.8, every principal ideal is compact in $IdR$.

In the same way, a subspace generated by a singleton set is compact in the complete lattice $SubV$ of all subspace of a vector space $V$

(2) Let $L = [0, 1]$ be the complete chain with the usual order. We consider the adjunction $\bigvee \dashv \downarrow$, where $\downarrow : L \to \mathcal{P}(L)(x \mapsto \downarrow x)$ and $\bigvee : \mathcal{P}(L) \to L(S \mapsto \bigvee S)$. $\downarrow^l = \bigvee$, and $\downarrow$ is 1-1 and preserves arbitrary meets. Hence $L$ has the $(\bigwedge, \downarrow)$-structure in $\mathcal{P}(L)$.

The singleton set $\{1\}$ is an atom in $\mathcal{P}(L)$ and $1 = \vee\{1\} = i^l(\{1\}) \in E_{\downarrow}(L)$, but 1 is not compact in $L$, hence $\downarrow$ does not preserve directed joins. In fact, if $D = \{1 - 1/n \mid n \in \mathbb{N}\}$, then $D$ is a directed subset of $L$ and $\downarrow (\bigvee_L D) = \downarrow 1 = L$, but $\bigvee_{\mathcal{P}(L)}\{\downarrow (1 - 1/n) \mid n \in \mathbb{N}\} = \bigcup\{\downarrow (1 - 1/n) \mid n \in \mathbb{N}\} = [0, 1) \neq L$.

LEMMA 2.10. *Let $L$ have a $(\bigwedge, f)$-structure in an atomistic complete lattice $H$. Then $x = \bigvee(\downarrow x \cap E_f(L))$ for every $x \in L$.*

*Proof.* From the definition of atomistic, $f(x) = \bigvee(\downarrow f(x) \cap A(H))$ for all $x \in L$. Since $f^l \dashv f$ and $f$ is order-embedding, $f^l$ preserves arbitrary joins and $f^l f = 1_L$ . Hence we have $x = f^l f(x) = f^l(\bigvee(\downarrow f(x) \cap A(H))) = \bigvee f^l(\downarrow f(x) \cap A(H)) = \bigvee(\downarrow x \cap E_f(L))$ for every $x \in L$. □

PROPOSITION 2.11. *Let $L$ have a $(\bigwedge, f)$-structure in an atomistic meet-continuous lattice $H$. If $x \ll y$ in $L$, then $x \leq \vee_L F \leq y$ for some $F \in Fin(\downarrow y \cap E_f(L))$.*

*Proof.* Suppose that $x \ll y$ in $L$. $y = \bigvee(\downarrow y \cap E_f(L))$, hence there is $F \in Fin(\downarrow y \cap E_f(L))$ such that $x \leq \vee_L F$, and we have $\vee_L F \leq \bigvee_L(\downarrow y \cap E_f(L)) \leq y$ since $F \subseteq \downarrow y \cap E_f(L) \subseteq \downarrow y$. □

The converse of the above proposition is not true in general.

EXAMPLE 2.12. In Example 2.9-(2), $A(\mathcal{P}(L)) = \{\{x\} \mid x \in L\}$, and $E_{\downarrow}(L) = \{\vee\{x\} \mid \{x\} \in A(\mathcal{P}(L))\} = \{x \mid \{x\} \in A(\mathcal{P}(L))\} = L$, hence we have $\downarrow 1 \cap E_{\downarrow}(L) = L \cap L = L$. For $F = \{1\} \in Fin(\downarrow 1 \cap E_{\downarrow}(L))$, $1 \leq \vee F \leq 1$, but $1 \not\ll 1$.

PROPOSITION 2.13. *Let $L$ have a $(\bigwedge, f)$-structure in an atomistic meet-continuous lattice $H$. If $f$ preserves directed joins, then the following are equivalent :*

(1) *$x \ll y$ in $L$*
(2) *$x \leq_L \vee_L F \leq_L y$ for some $F \in Fin(\downarrow y \cap E_f(L))$.*

*Proof.* It is clear from Proposition 2.3 and 2.11. □

EXAMPLE 2.14. The complete lattice $IdR$ of a ring has a $\bigwedge$-structure in $\mathcal{P}(R)$. For every directed subset $\mathcal{D}$ of $IdR$, $\bigvee_{\mathcal{P}(R)} \mathcal{D} = \bigcup \mathcal{D} \in IdR$, and every principal ideal is compact. Hence we have that $I \ll J$ in $IdR$ if and only if $I \leq \vee\mathcal{F} \leq J$ for some $\mathcal{F} \in Fin(\{(x) \mid x \in R\})$, where $(x)$ is the principal ideal of $R$ generated by $x \in R$.

In the same way, let $SubV$ be the complete lattice of all subspaces of a vector space $V$. Then $U \ll V$ in $SubV$ if and only if $U \leq \vee\mathcal{F} \leq V$ for some $\mathcal{F} \in Fin(\{[v] \mid v \in V\})$, where $[v]$ is the subspace of $V$ generated by $v \in V$.

Proposition 2.15. *Let $L$ have a $(\bigwedge, f)$-structure in an atomistic meet-continuous lattice $H$. If $f$ preserves directed joins, then $L$ is continuous.*

*Proof.* Suppose that $f$ preserves directed joins, and let $x \in L$. Then for every $a \in \downarrow f(x) \cap A(H)$, $a \ll a \leq f(x)$ by , and $a \ll f(x)$. Hence $\downarrow f(x) \cap A(H) \subseteq \downarrow_w f(x)$, and it implies that $\downarrow x \cap E_f(L) = f^l(\downarrow f(x) \cap A(H)) \subseteq f^l(\downarrow_w f(x))$. Since $f^l(\downarrow_w f(x)) \subseteq \downarrow_w x$, $\downarrow x \cap E_f(L) \subseteq \downarrow_w x$, and we have $x = \bigvee(\downarrow x \cap E_f(L)) \leq \bigvee(\downarrow_w x) \leq x$ . Hence $x = \bigvee(\downarrow_w x)$, and $L$ is continuous. □

Example 2.16. The complete lattices $IdR$ and $SubV$ are continuous.

Proposition 2.17. *Let $L$ have a $(\bigwedge, f)$-structure in an atomistic meet-continuous lattice $H$. If every element of $E_f(L)$ is compact, then $L$ is algebraic.*

*Proof.* Suppose that every element of $E_f(L)$ is compact. Then $E_f(L) \subseteq K(L)$, and

$\downarrow x \cap E_f(L) \subseteq \downarrow x$ Thus we have $x = \bigvee(\downarrow x \cap K(L))$. □

The converse of the above proposition is not true. In fact, $L$ in Example 2.9-(2) is algebraic, but 1 is not compact.

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