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A FUNCTIONAL EQUATION ON HYPERPLANES PASSING THROUGH THE ORIGIN

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ABSTRACT. In this paper, we obtain the general solution and the stability of the multi-dimensional Cauchy's functional equation

$$f(x_1 + y_1, \cdots, x_n + y_n) = f(x_1, \cdots, x_n) + f(y_1, \cdots, y_n).$$

The function f given by $f(x_1, \cdots, x_n) = a_1x_1 + \cdots + a_nx_n$ is a solution of the above functional equation.

1. Introduction

Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be real numbers not all equal to 0. Then the set consisting of all vectors $(x_1, \cdots, x_n)$ in $\mathbb{R}^n$ such that

$$\alpha_1x_1 + \alpha_2x_2 + \cdots + \alpha_nx_n = c \tag{1.1}$$

for c a constant is a subspace of $\mathbb{R}^n$ called a hyperplane. Thus, for example, lines are hyperplanes in $\mathbb{R}^2$ and planes are hyperplanes in $\mathbb{R}^3$. If $c = 0$ in (1.1), then the equation simplifies to

$$\alpha_1x_1 + \alpha_2x_2 + \cdots + \alpha_nx_n = 0$$

and we say that the hyperplane **passes through the origin** or the **linear** hyperplane.

More generally, a linear hyperplane is any codimension-1 vector subspace of a vector space. Equivalently, a linear hyperplane V in a vector space W is any subspace such that W/V is one-dimensional. Equivalently, a linear hyperplane is the linear transformation kernel of any nonzero linear map from the vector space to the underlying field.

In this paper, let X and Y be real vector spaces. For a mapping $f : X^n \to Y$, consider the multi-dimensional Cauchy's functional equation:

$$f(x_1 + y_1, \cdots, x_n + y_n) = f(x_1, \cdots, x_n) + f(y_1, \cdots, y_n) \tag{1.2}$$

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When $X = Y = \mathbb{R}$, the function $f : \mathbb{R}^n \to \mathbb{R}$ given by

$$f(x_1, \cdots, x_n) = a_1x_1 + \cdots + a_nx_n$$

is a solution of (1.2).

For a mapping $g : X \to Y$, consider the Cauchy's functional equation:

$$g(x + y) = g(x) + g(y). \tag{1.3}$$

Recently, the authors investigeted various functional equations ([2], [3], [4]). In this paper, we investigate the relation between (1.2) and (1.3). And we find out the general solution and prove the generalized Hyers-Ulam stability of (1.2).

2. Results

The multi-dimensional vector-variable Cauchy's functional equation (1.2) induces the Cauchy's functional equation (1.3) as follows.

THEOREM 2.1. Let $f : X^n \to Y$ be a mapping satisfying (1.2) and let $g : X \to Y$ be the mapping given by

$$g(x) := f(x, \cdots, x) \tag{2.1}$$

for all $x \in X$, then g satisfies (1.3).

Proof. By (1.2) and (2.1),

$$\begin{aligned} g(x + y) &= 2f(x + y, \cdots, x + y) \\ &= f(x, \cdots, x) + f(y, \cdots, y) \\ &= g(x) + g(y) \end{aligned}$$

for all $x, y \in X$. □

The Cauchy's functional equation (1.3) induces the multi-dimensional Cauchy's functional equation (1.2) with an additional condition.

THEOREM 2.2. Let $a_1, \cdots, a_n \in \mathbb{R}$ and $g : X \to Y$ be a mapping satisfying (1.3). If $f : X^n \to Y$ is the mapping given by

$$f(x_1, \cdots, x_n) := a_1g(x_1) + \cdots + a_ng(x_n) \tag{2.2}$$

for all $x_1, \cdots, x_n \in X$, then f satisfies (1.2). Furthermore, (2.1) holds if

$$a_1 + \cdots + a_n = 1.$$

Proof. By (1.3) and (2.2),

$$\begin{aligned} &f(x_1+y_1,\cdots,x_n+y_n)\\ &\quad= a_1g(x_1+y_1)+\cdots+a_ng(x_n+y_n)\\ &\quad= a_1[g(x_1)+g(y_1)]+\cdots+a_n[g(x_n)+g(y_n)]\\ &\quad= [a_1g(x_1)+\cdots+a_ng(x_n)]+[a_1g(y_1)+\cdots+a_ng(y_n)]\\ &\quad= f(x_1,\cdots,x_n)+f(y_1,\cdots,y_n) \end{aligned}$$

for all $x_1,\cdots,x_n,y_1,\cdots,y_n \in X$. Furthermore, if $a_1+\cdots+a_n=1$,

$$f(x,\cdots,x)=a_1g(x)+\cdots+a_ng(x)=g(x)$$

for all $x \in X$. $\square$

EXAMPLE 2.1. *Let X be the space of $m \times m$ real matrices. Consider the mapping $g : X \to \mathbb{R}$ given by $g(A) := tr(A)$ =the trace of A for all $A \in X$. For $a_1,\cdots,a_n \in \mathbb{R}$, consider the function $f : X^n \to \mathbb{R}$ given by*

$$f(A_1,\cdots,A_n)=\sum_{i=1}^{n} a_i\ tr(A_i)$$

for all $A_1,\cdots,A_n \in X$. By Theorem 2.2, the function f satisfies (1.2).

In the following theorem, we find out the general solution of the multi-dimensional Cauchy's functional equation (1.2).

THEOREM 2.3. *A mapping $f : X^n \to Y$ satisfies* (1.2) *if and only if there exist additive mappings $A_1,\cdots,A_n : X \to Y$ such that*

$$f(x_1,\cdots,x_n)=A_1(x_1)+\cdots+A_n(x_n)$$

for all $x_1,\cdots,x_n \in X$.

Proof. We first assume that f is a solution of (1.2). Define $A_1,\cdots,A_n : X \to Y$ by

$$A_1(x) := f(x,0,\cdots,0),\cdots,A_n(x) := f(0,\cdots,0,x)$$

for all $x_1,\cdots,x_n \in X$. One can easily verify that $A_1,\cdots,A_n$ are additive mappings.

Letting $y_2=-x_2,\cdots,y_n=-x_n$ and $y_1=x_1$ in (1.2),

$$2f(x_1,0,\cdots,0)=f(x_1,\cdots,x_n)+f(x_1,-x_2,\cdots,-x_n)$$

for all $x_1, \cdots, x_n \in X$. Putting $y_2 = x_2, \cdots, y_n = x_n$ and $y_1 = -x_1$ in (1.2),

$$f(0, 2x_2, \cdots, 2x_n) = f(x_1, \cdots, x_n) + f(-x_1, x_2, \cdots, x_n)$$

for all $x_1, \cdots, x_n \in X$. Setting $y_1 = -x_1, \cdots, y_n = -x_n$ in (1.2),

$$f(-x_1, \cdots, -x_n) = -f(x_1, \cdots, x_n)$$

for all $x_1, \cdots, x_n \in X$. By the above three equalities,

$$f(x_1, \cdots, x_n) = f(x_1, 0, \cdots, 0) + \frac{1}{2} f(0, 2x_2, \cdots, 2x_n) \tag{2.3}$$

for all $x_1, \cdots, x_n \in X$. Letting $y_2 = x_2$, $y_3 = -x_3, \cdots, y_n = -x_n$ and $x_1 = y_1 = 0$ in (1.2),

$$2f(0, x_2, 0, \cdots, 0) = f(0, x_2, x_3, \cdots, x_n) + f(0, x_2, -x_3, \cdots, -x_n)$$

for all $x_2, \cdots, x_n \in X$. Putting $y_2 = -x_2$, $y_3 = x_3, \cdots, y_n = x_n$ and $x_1 = y_1 = 0$ in (1.2),

$$f(0, 0, 2x_3, \cdots, 2x_n) = f(0, x_2, x_3, \cdots, x_n) + f(0, -x_2, x_3, \cdots, x_n)$$

for all $x_2, \cdots, x_n \in X$. By the above two equalities,

$$f(0, x_2, \cdots, x_n) = f(0, x_2, 0, \cdots, 0) + \frac{1}{2} f(0, 0, 2x_3, \cdots, 2x_n) \tag{2.4}$$

for all $x_2, \cdots, x_n \in X$. By (2.3) and (2.4),

$$\begin{aligned} f(x_1, \cdots, x_n) &= f(x_1, 0, \cdots, 0) + f(0, x_2, 0, \cdots, 0) \\ &\quad + \frac{1}{2^2} f(0, 0, 2^2 x_3, \cdots, 2^2 x_n), \end{aligned}$$

for all $x_1, \cdots, x_n \in X$. Continuing this process, one can obtain that

$$\begin{aligned} f(x_1, \cdots, x_n) &= f(x_1, 0, \cdots, 0) + \cdots + \frac{1}{2^{n-1}} f(0, \cdots, 0, 2^{n-1} x_n) \\ &= f(x_1, 0, \cdots, 0) + \cdots + f(0, \cdots, 0, x_n) \\ &= A_1(x_1) + \cdots + A_n(x_n) \end{aligned}$$

for all $x_1, \cdots, x_n \in X$.

Conversely, we assume that there exist additive mappings $A_1, \cdots, A_n : X^n \to Y$ such that

$$f(x_1, \cdots, x_n) = A_1(x_1) + \cdots + A_n(x_n)$$

for all $x_1, \cdots, x_n \in X$. Since $A_1, \cdots, A_n$ are additive,

$$\begin{aligned} &f(x_1 + y_1, \cdots, x_n + y_n) \\ &\quad = 2A_1(x_1 + y_1) + \cdots + 2A_n(x_n + y_n) \\ &\quad = A_1(x_1) + A_1(y_1) + \cdots + A_n(x_n) + A_n(y_n)) \\ &\quad = f(x_1, \cdots, x_n) + f(y_1, \cdots, y_n) \end{aligned}$$

for all $x_1, \cdots, x_n, y_1, \cdots, y_n \in X$. $\square$

Let Y be complete and let $\varphi : X^{2n} \to [0, \infty)$ be a function satisfying

$$\begin{aligned} (2.5) \qquad &\tilde{\varphi}(x_1, \cdots, x_n, y_1, \cdots, y_n) \\ &:= \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} \varphi(2^j x_1, \cdots, 2^j x_n, -2^j x_1, \cdots, -2^j x_n) < \infty \end{aligned}$$

for all $x_1, \cdots, x_n, y_1, \cdots, y_n \in X$.

THEOREM 2.4. Let $f : X^n \to Y$ be a mapping such that

$$\begin{aligned} (2.6) \qquad &\|f(x_1 + y_2, \cdots, x_n + y_n) - f(x_1, \cdots, x_n) - f(y_1, \cdots, y_n)\| \\ &\leq \varphi(x_1, \cdots, x_n, y_1, \cdots, y_n) \end{aligned}$$

for all $x_1, \cdots, x_n, y_1, \cdots, y_n \in X$. Then there exists a unique multi-dimensional Cauchy's mapping $F : X^n \to Y$ such that

$$\begin{aligned} (2.7) &\|f(x_1, \cdots, x_n) - F(x_1, \cdots, x_n)\| \\ &\leq \tilde{\varphi}(x_1, \cdots, x_n, x_1, \cdots, x_n) + \tilde{\varphi}(-x_1, \cdots, -x_n, -2x_1, \cdots, -2x_n) \end{aligned}$$

for all $x_1, \cdots, x_n \in X$.

Proof. Letting $y_1 = -x_1, \cdots, y_n = -x_n$ in (2.6), we have

$$\|f(x_1, \cdots, x_n) + f(-x_1, \cdots, -x_n)\| \leq \varphi(x_1, \cdots, x_n, -x_1, \cdots, -x_n)$$

for all $x_1, \cdots, x_n \in X$. Replacing $x_1, \cdots, x_n, y_1, \cdots, y_n$ by $-x_1, \cdots, -x_n, 2x_1, \cdots, 2x_n$ respectively in (2.6), we have

$$\begin{aligned} &\|f(x_1, \cdots, x_n) - f(-x_1, \cdots, -x_n) - f(2x_1, \cdots, 2x_n)\| \\ &\quad \leq \varphi(-x_1, \cdots, -x_n, 2x_1, \cdots, 2x_n) \end{aligned}$$

for all $x_1, \cdots, x_n \in X$. By the above two inequalities,

$$\begin{aligned} &\|f(x_1, \cdots, x_n) - \frac{1}{2} f(2x_1, \cdots, 2x_n)\| \\ &\leq \frac{1}{2} \big[\varphi(x_1, \cdots, x_n, -x_1, \cdots, -x_n) + \varphi(-x_1, \cdots, -x_n, 2x_1, \cdots, 2x_n)\big] \end{aligned}$$

for all $x_1, \cdots, x_n \in X$. Thus we obtain

$$\begin{aligned}
&\left\| \frac{1}{2^j} f(2^j x_1, \cdots, 2^j x_n) - \frac{1}{2^{j+1}} f(2^{j+1} x_1, \cdots, 2^{j+1} x_n) \right\| \\
&\quad \le \frac{1}{2^{j+1}} \big[\varphi(2^j x_1, \cdots, 2^j x_n, -2^j x_1, \cdots, -2^j x_n) \\
&\qquad\qquad + \varphi(-2^j x_1, \cdots, -2^j x_n, 2^{j+1} x_1, \cdots, 2^{j+1} x_n) \big]
\end{aligned}$$

for all $x_1, \cdots, x_n \in X$ and all j. For given integers $l, m (0 \le l < m)$, we get

$$\begin{aligned}
&\left\| \frac{1}{2^l} f(2^l x_1, \cdots, 2^l x_n) - \frac{1}{2^m} f(2^m x_1, \cdots, 2^m x_n) \right\| \\
&\quad \le \sum_{j=l}^{m-1} \frac{1}{2^{j+1}} [\varphi(2^j x_1, \cdots, 2^j x_n, -2^j x_1, \cdots, -2^j x_n) \\
&\qquad\qquad + \varphi(-2^j x_1, \cdots, -2^j x_n, 2^{j+1} x_1, \cdots, 2^{j+1} x_n)]
\end{aligned} \tag{2.8}$$

for all $x_1, \cdots, x_n \in X$. By (2.8), the sequence $\{\frac{1}{2^j} f(2^j x_1, \cdots, 2^j x_n)\}$ is a Cauchy sequence for all $x_1, \cdots, x_n \in X$. Since Y is complete, the sequence $\{\frac{1}{2^j} f(2^j x_1, \cdots, 2^j x_n)\}$ converges for all $x_1, \cdots, x_n \in X$. Define $F : X^n \to Y$ by

$$F(x_1, \cdots, x_n) := \lim_{j \to \infty} \frac{1}{2^j} f(2^j x_1, \cdots, 2^j x_n)$$

for all $x_1, \cdots, x_n \in X$. By (2.6), we have

$$\begin{aligned}
&\| f(2^j(x_1 + y_1), \cdots, 2^j(x_n + y_n)) - f(2^j x_1, \cdots, 2^j x_n) \\
&\quad - f(2^j y_1, \cdots, 2^j y_n) \| \le \varphi(2^j x_1, \cdots, 2^j x_n, 2^j y_1, \cdots, 2^j y_n)
\end{aligned}$$

for all $x_1, \cdots, x_n, y_1, \cdots, y_n \in X$ and all j. Letting $j \to \infty$ and using (2.5), we see that F satisfies (1.2). Setting $l = 0$ and taking $m \to \infty$ in (2.8), one can obtain the inequality (2.7). If $G : X^n \to Y$ is another multi-dimensional additive mapping satisfying (2.7), we obtain

$$\begin{aligned}
&\|F(x_1,\cdots,x_n)-G(x_1,\cdots,x_n)\|\\
&\quad=\frac{1}{2^n}\|F(2^nx_1,\cdots,2^nx_n)-G(2^nx_1,\cdots,2^nx_n)\|\\
&\quad\le\frac{1}{2^n}\|F(2^nx_1,\cdots,2^nx_n)-f(2^nx_1,\cdots,23^nx_n)\|\\
&\qquad+\frac{1}{2^n}\|f(2^nx_1,\cdots,2^nx_n)-G(2^nx_1,\cdots,2^nx_n)\|\\
&\quad\le\frac{2}{2^n}[\tilde{\varphi}(2^nx_1,\cdots,2^nx_n,2^nx_1,\cdots,2^nx_n)\\
&\qquad+\tilde{\varphi}(-2^nx_1,\cdots,-2^nx_n,-2^{n+1}x_1,\cdots,-2^{n+1}x_n)]\\
&\quad\to 0\ \text{ as }\ n\to\infty
\end{aligned}$$

for all $x_1,\cdots,x_n\in X$. Hence the mapping F is the unique multi-dimensional additive mapping, as desired. □

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