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ON THE EXISTENCE OF THE THIRD SOLUTION OF THE NONLINEAR BIHARMONIC EQUATION WITH DIRICHLET BOUNDARY CONDITION

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ABSTRACT. We are concerned with the multiplicity of solutions of the nonlinear biharmonic equation with Dirichlet boundary condition, $\Delta^2 u + c\Delta u = g(u)$, in Ω, where $c \in R$ and Δ^2 denotes the biharmonic operator. We show that there exists at least three solutions of the above problem under the suitable condition of $g(u)$.

1. Introduction

Let Ω be a smooth bounded region in R^n with smooth boundary $\partial\Omega$. Let $\lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots$ be the eigenvalues of $-\Delta$ with Dirichlet boundary condition in Ω. In this paper we are concerned with the multiplicity of solutions of the nonlinear biharmonic equation with Dirichlet boundary condition

$$\Delta^2 u + c\Delta u = g(u) \qquad \text{in } \Omega, \tag{1.1}$$
$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega,$$

where g is a differentiable function from R to R such that $g(0) = 0$, $c \in R$ and Δ^2 denotes the biharmonic operator. Let

$$g'(\infty) = \lim_{|u|\to\infty} \frac{g(u)}{u} \in R.$$

The problem (1.1) was studied by Choi and Jung in [5]. The authors proved that (1.1) has at least two solutions by a variation of linking theorem. The authors also proved in [7] that the problem

$$\Delta^2 u + c\Delta u = bu^+ + s \qquad \text{in } \Omega, \tag{1.2}$$

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$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega$$

has at least two solutions by a variational reduction method when $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ or $c < \lambda_1$, $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)$. This type problem arises in the study of travelling waves in a suspension bridge ([8], [10]) or the study of the static deflection of an elastic plate in a fluid. The following is the main result of this paper.

Theorem 1.1. *Assume that* $\lambda_i < c < \lambda_{i+1}$, $\lambda_{i+1}(\lambda_{i+1} - c) < \lambda_k(\lambda_k - c) < g'(\infty) < \lambda_{k+1}(\lambda_{k+1} - c)$, $\lambda_{k+m}(\lambda_{k+m} - c) < g'(0) < \lambda_{k+m+1}(\lambda_{k+m+1} - c)$ *and* $g'(t) \leq \gamma < \lambda_{k+m+1}(\lambda_{k+m+1} - c)$, *where* $m \geq 1$, $k > i + 1$ *and* $\gamma \in R$. *Then problem (1.1) has at least three solutions.*

Theorem 1.2. *Assume that* $\lambda_i < c < \lambda_{i+1}$, $\lambda_{i+1}(\lambda_{i+1} - c) < \lambda_k(\lambda_k - c) < g'(0) < \lambda_{k+1}(\lambda_{k+1} - c)$, $\lambda_{k+m}(\lambda_{k+m} - c) < g'(\infty) < \lambda_{k+m+1}(\lambda_{k+m+1} - c)$ *and* $g'(t) \leq \gamma < \lambda_{k+m+1}(\lambda_{k+m+1} - c)$, *where* $m \geq 1$, $k > i + 1$ *and* $\gamma \in R$. *Then problem (1.1) has at least three solutions.*

In section 2 we recall a Linking Scale Theorem which will play a crucial role in our argument. In section 3 we define a Banach space H spanned by eigenfunctions of $\Delta^2 + c\Delta$ with Dirichlet boundary condition which can be applied in the linking scale theorem. In section 4 we prove Theorem 1.1 and Theorem 1.2.

2. Linking scale theorem

Definition 2.1. Let X be a Hilbert space, $Y \subset X$, $rho > 0$ and $e \in X \backslash Y$, $e \neq 0$. Set:

$$B_\rho(Y) = \{x \in Y | \|x\|_X \leq \rho\},$$
$$S_\rho(Y) = \{x \in Y | \|x\|_X = \rho\},$$
$$\Delta_\rho(e, Y) = \{\sigma e + v | \sigma \geq 0, v \in Y, \|\sigma e + v\|_X \leq \rho\},$$
$$\Sigma_\rho(e, Y) = \{\sigma e + v | \sigma \geq 0, v \in Y, \|\sigma e + v\|_X = \rho\} \cup \{v | v \in Y, \|v\|_X \leq \rho\}.$$

Now we recall a theorem of existence of three solutions which is linking scale theorem.

Theorem 2.1. *(Linking scale theorem) Let* X *be an Hilbert space, which is topological direct sum of the four subspaces* X_0, X_1, X_2 *and* X_3. *Let* $F \in C^1(X, R)$. *Moreover assume:*
(a) $dim X_i < +\infty$ *for* $i = 0, 1, 2$;
(b) *there exist* $\rho > 0$, $R > 0$ *and* $e \in X_2$, $e \neq 0$ *such that;*

$$\rho < R \qquad \text{and} \qquad \sup_{S_\rho(X_0 \oplus X_1 \oplus X_2)} F < \inf_{\Sigma_R(e, X_3)} F;$$

(c) there exist $\rho' > 0$, $R' > 0$ and $e' \in X_1$, $e' \neq 0$ such that:

$$\rho' < R' \qquad \text{and} \qquad \sup_{S_{\rho'}(X_0 \oplus X_1)} F \leq \inf_{\Sigma_{R'}(e', X_2 \oplus X_3)} F;$$

(d) $R \leq R' (\Rightarrow \Delta_R(e, X_3) \subset \Sigma_{R'}(e', X_2 \oplus X_3))$;
(e) $-\infty < a = \inf_{\Delta_{R'}(e, X_2 \oplus X_3)} F$;
(f) $(P.S.)_c$ holds for any $c \in [a, b]$ where $b = \sup_{B_\rho(X_0 \oplus X_1 \oplus X_2)} F$.
Then there exist three critical levels c_1, c_2 and c_3 for the functional F such that:

$$a \leq c_3 \leq \sup_{S_{\rho'}(X_0 \oplus X_1)} F < \inf_{\Sigma_{R'}(e', X_2 \oplus X_3)} F \leq \inf_{\Delta_R(e, X_3)} F \leq c_2$$

$$\leq \sup_{S_\rho(X_0 \oplus X_1 \oplus X_2)} F < \inf_{\Sigma_R(e. X_3)} F \leq c_1 \leq b.$$

3. Variational formulation

Let $\lambda_k (k = 1, 2, \ldots)$ denote the eigenvalues and $\phi_k (k = 1, 2, \ldots)$ the corresponding eigenfunctions, suitably normalized with respect to $L^2(\Omega)$ inner product, of the eigenvalue problem $\Delta u + \lambda u = 0$ in Ω, with the Dirichlet boundary condition, where each eigenvalue λ_k is repeated as often as its multiplicity. We recall that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots, \lambda_i \to +\infty$ and that $\phi_1(x) > 0$ for $x \in \Omega$. The eigenvalue problem $\Delta^2 u + c\Delta u = \mu u$ in Ω with the Dirichlet boundary condition $u = 0$, $\Delta u = 0$ on $\partial\Omega$, has infinitely many eigenvalues $\lambda_k(\lambda_k - c)$, $k = 1, 2, \ldots$, and corresponding eigenfunctions $\phi_k(x)$. The set of functions $\{\phi_k\}$ is an orthogonal base for $W_0^{1,2}(\Omega)$. Let us denote an element u of $W_0^{1,2}(\Omega)$ as $u = \sum h_k \phi_k, \sum h_k^2 < \infty$. Let c be not an eigenvalue of $-\Delta$ and define a subspace E of $W_0^{1,2}(\Omega)$ as follows

$$E = \{u \in W_0^{1,2}(\Omega) : \sum |\lambda_k(\lambda_k - c)| h_k^2 < \infty\}.$$

Then this is a complete normed space with a norm

$$|||u||| = [\sum |\lambda_k(\lambda_k - c)| h_k^2]^{\frac{1}{2}}.$$

We need the following some properties which are proved in [6, 7]. Since $\lambda_k \to +\infty$ and c is fixed, we have:

PROPOSITION 3.1. *Let c be not an eigenvalue of $-\Delta$ with the Dirichlet boundary condition. Then we have*
(i) $(\Delta^2 u + c\Delta)u \in E$ implies $u \in E$.

(ii) $|||u||| \geq C\|u\|_{L^2(\Omega)}$, *for some* $C > 0$.
(iii) $\|u\|_{L^2(\Omega)} = 0$ *if and only if* $|||u||| = 0$.

PROPOSITION 3.2. *Assume that* $g : E \to R$ *satisfies the assumptions of Theorem 1.1. Then all solutions in* $L^2(\Omega)$ *of*

$$\Delta^2 u + c\Delta u = g(u) \qquad in\ L^2(\Omega)$$

belong to E.

Proof. Let $g(u) = \sum h_k\phi_k \in L^2(\Omega)$. Then

$$(\Delta^2 + c\Delta)^{-1}(g(u)) = \sum \frac{1}{\lambda_k(\lambda_k - c)} h_k\phi_k.$$

Hence we have

$$|||(\Delta^2 + c\Delta)^{-1}g(u)|||^2 = \sum |\lambda_k(\lambda_k - c)| \frac{1}{\lambda_k(\lambda_k - c))^2} h_k^2 \leq C \sum h_k^2$$

for some $C > 0$, which means that

$$|||(\Delta^2 + c\Delta)^{-1}g(u)||| \leq C_1\|u\|_{L^2(\Omega)}.$$

□

With the aid of Proposition 3.2 it is enough that we investigate the existence of solutions of (1.1) in the subspace E of $L^2(\Omega)$. Let $I : E \to R$ be the functional defined by,

$$I(u) = \int_\Omega \frac{1}{2}|\Delta u|^2 - \frac{c}{2}|\nabla u|^2 - G(u), \tag{3.1}$$

where $G(s) = \int_0^s g(\sigma)d\sigma$. Under the assumptions of Theorem 1.1, $I(u)$ is well defined. By the following Proposition, I is of class C^1 and the weak solutions of (1.1) coincide with the critical points of $I(u)$.

PROPOSITION 3.3. *Assume that* $g(u)$ *satisfies the assumptions of Theorem 1.1. Then* $I(u)$ *is continuous and Frèchet differentiable in* E *and*

$$DI(u)(h) = \int_\Omega \Delta u \cdot \Delta h - c\nabla u \cdot \nabla h - g(u)h \tag{3.2}$$

for $h \in X$. *Moreover* $\int_\Omega G(u)dx$ *is* C^1 *with respect to* u. *Thus* $I \in C^1$.

Proof. Let $u \in E$. First we will prove that $I(u)$ is continuous. We consider

$$\begin{aligned} I(u+v) - I(u) &= \int_\Omega [\frac{1}{2}|\Delta(u+v)|^2 - \frac{c}{2}|\nabla(u+v)|^2 - G(u+v)] \\ &\quad - \int_\Omega [\frac{1}{2}|\Delta u|^2 - \frac{c}{2}|\nabla u|^2 - G(u)] \\ &= \int_\Omega [u \cdot (\Delta^2 v + c\Delta v) + \frac{1}{2} v \cdot (\Delta^2 v + c\Delta v) \\ &\quad - (G(u+v) - G(u))]. \end{aligned}$$

Let $u = \sum h_k \phi_k$, $v = \sum \tilde{h}_k \phi_k$. Then we have

$$|\int_\Omega u \cdot (\Delta^2 v + c\Delta v) dx| = |\sum \lambda_k(\lambda_k - c) h_k \tilde{h}_k| \le |||u||| \cdot |||v|||$$

$$|\int_\Omega v \cdot (\Delta^2 v + c\Delta v) dx| = |\sum \lambda_k(\lambda_k - c) \tilde{h}_k^2| \le |||v|||^2.$$

On the other hand, by Mean Value Theorem and $g'(t) \le \gamma$, we have

$$\begin{aligned} |G(u+v) - G(u)| &= |\int_0^{u+v} g(s)ds - \int_0^u g(s)ds| \\ &\le \gamma |v|(|u| + |v|) \end{aligned}$$

Hence

$$|\int_\Omega [G(u+v) - G(u)] dx| \le C\gamma |||v|||(|||u||| + |||v|||).$$

With the above results, we see that $I(u)$ is continuous at u. To prove that $I(u)$ is Fréchet differentiable at $u \in E$, we compute

$$\begin{aligned} |I(u+v) &- I(u) - DI(u)v| \\ &= |\int_\Omega \frac{1}{2} v(\Delta^2 v + c\Delta v) - G(u+v) + G(u) - g(u)v| \\ &\le \frac{1}{2}|||v|||^2 + C\gamma |||v|||^2, \end{aligned}$$

since $|G(u+v) - G(u) - g(u)v| = |\int_u^{u+v} g(s)ds - g(u)v| \le \gamma v^2$. □

Let Z_2 act on E orthogonally. Then E has two invariant orthogonal subspaces Fix_{Z_2} and $Fix_{Z_2}^{\perp}$. Let us set

$$H = Fix_{Z_2}^{\perp}.$$

The Z_2 action has the representation $u \mapsto -u$, $\forall u \in H$. Thus Z_2 acts freely on the invariant subspace H. We note that H is a closed invariant linear subspace of E compactly embedded in $L^2(\Omega)$. It is easily checked that $\Delta^2 + c\Delta$ and g are equivariant on H, so I is invariant on H. Moreover $(\Delta^2 + c\Delta)(H) \subseteq H$, $\Delta^2 + c\Delta : H \to H$ is an isomorphism and $DI(H) \subseteq H$. Therefore critical points on H are critical points on E.

4. Proof of Theorem 1.1 and Theorem 1.2.

Here we let $\lambda_i < c < \lambda_{i+1}$. First, we consider the case $\lambda_{i+1}(\lambda_{i+1}-c) < \lambda_k(\lambda_k - c) < g'(\infty) < \lambda_{k+1}(\lambda_{k+1} - c)$, $\lambda_{k+m}(\lambda_{k+m} - c) < g'(0) < \lambda_{k+m+1}(\lambda_{k+m+1}-c)$ and $g'(t) \leq \gamma < \lambda_{k+m+1}(\lambda_{k+m+1}-c)$, where $m \geq 1$ and $k > i+1$. Let H_k be the subspace of H spanned by $\phi_1, \ldots, \phi_k$ whose eigenvalues are $\lambda_1(\lambda_1 - c), \ldots, \lambda_k(\lambda_k - c)$. Let $H_k^\perp$ be the orthogonal complement of H_k in H. Let $r = \frac{1}{2}\{\lambda_k(\lambda_k - c) + \lambda_{k+1}(\lambda_{k+1} - c)\}$ and let $L : H \to H$ be the linear continuous operator such that

$$(Lu, v) = \int_\Omega (\Delta^2 u + c\Delta u) \cdot v dx - r \int_\Omega uv dx.$$

Then L is symmetric, bijective and equivariant. The spaces H_k, $H_k^\perp$ are the negative space of L and the positive space of L. Moreover, there exists $\nu > 0$ such that

$$\forall u \in H_k \quad : \qquad (Lu, u) \leq (\lambda_k(\lambda_k - r)) \int_\Omega u^2 dx \leq -\nu |\|u\||^2,$$

$$\forall u \in H_k^\perp \quad : \qquad (Lu, u) \geq (\lambda_{k+1}(\lambda_{k+1} - c)) \int_\Omega u^2 dx \geq \nu |\|u\||^2.$$

We can write

$$I(u) = \frac{1}{2}(Lu, u) - \psi(u),$$

where

$$\psi(u) = \int_\Omega [G(u) - \frac{1}{2} r u^2] dx.$$

Since H is compactly embedded in L^2, the map $D\psi : X \to X$ is compact.

Lemma 4.1. *Assume that $g(u)$ satisfies the assumptions of Theorem 1.1. Then $I(u)$ satisfies the $(P.S.)_M$ condition for any $M \in R$.*

Proof. Let (u_n) be a sequence in H with $DI(u_n) \to 0$ and $I(u_n) \to M$. Since L is an isomorphism and $D\psi$ is compact, it is sufficient to show that

(u_n) is bounded in H. By contradiction, we assume that $|||u_n||| \to +\infty$. Let us take $a, b \in R$ with

$$\lambda_k(\lambda_k - c) < a < \lim_{|s|\to\infty} \frac{g(s)}{s} < b < \lambda_{k+1}(\lambda_{k+1} - c)$$

and define $g_r : R \to R$ by

$$g_r(s) = g(s) - rs.$$

Set $\alpha = a - r$ and $\beta = b - r$, so that

$$\alpha < \lim_{|s|\to\infty} \frac{g_r(s)}{s} < \beta.$$

Let $g_r(s) = \eta_r(s) + \gamma_r(s)s$ with

$$\gamma_r(s) = \begin{cases} \min\{\max\{\frac{g_r(s)}{s}, \alpha\}, \beta\} & \text{if } s \neq 0, \\ \min\{\max\{g'(0) - r, \alpha\}, \beta\} & \text{if } s = 0. \end{cases}$$

Then γ_r is a Borel function with $\alpha \le \gamma_r(s) \le \beta$ for every $s \in R$ and $\eta_r \in C_c(R)$. Let $v_n = \frac{u_n}{|||u_n|||}$. Up to a subsequence, we have $v_n \to v$ in H and $\gamma_r(u_n) \rightharpoonup \gamma'$ in $L^\infty(\Omega)$ with $\alpha \le \gamma' \le \beta$ a.e. in Ω. Moreover

$$\frac{\eta_r(u_n)}{|||u_n|||} \to 0 \qquad \text{in } L^\infty(\Omega).$$

Since $DI(u_n) \to 0$, we get

$$\frac{DI(u_n)u_n}{|||u_n|||^2} = (Lv_n, v_n) - \int_\Omega \frac{\eta_r(u_n)}{|||u_n|||} v_n - \int_\Omega \gamma_r(u_n) v_n^2 \longrightarrow 0.$$

Let $P^+ : H \to H_k^\perp$ and $P^- : H \to H_k$ denote the orthogonal projections. Since $P^+v_n - P^-v_n$ is bounded in H, we have

$$(LP^+v_n, P^+v_n) - (LP^-v_n, P^-v_n) - \int_\Omega \tfrac{\eta_r(u_n)}{|||u_n|||}(P^+v_n - P^-v_n)dx$$
$$- \int_\Omega \gamma_r(u_n)v_n(P^+v_n - P^-v_n)dx \longrightarrow 0.$$

Since $P^+v_n - P^-v_n \to P^+v - P^-v$ in H, we get

$$\nu \le \int_\Omega \gamma' v(P^+v - P^-v)dx.$$

Hence $v \neq 0$. On the other hand, we also have

$$(Lv_n, P^+v - P^-v) - \int_\Omega \frac{\eta_r(u_n)}{|||u_n|||}(P^+v - P^-v)dx$$
$$- \int_\Omega \gamma_r(u_n)v_n(P^+v - P^-v)dx \longrightarrow 0,$$

so that

$$(LP^+v, P^+v) - (LP^-v, P^-v) - \int_\Omega \gamma'(P^+v)^2 dx + \int_\Omega \gamma'(P^-v)^2 dx$$
$$= (Lv, P^+v - P^-v) - \int_\Omega \gamma'(P^+v + P^-v)(P^+v - P^-v)dx = 0.$$

It follows that

$$(\lambda_{k+1}(\lambda_{k+1}-c)-r-\beta)\int_\Omega (P^+v)^2 dx + (r+\alpha-\lambda_k(\lambda_k-c))\int_\Omega (P^-v)^2 dx \le 0.$$

Thus $P^+v = P^-v = 0$, which gives a contradiction. □

LEMMA 4.2. *Under the same assumptions of Theorem 1.1, The function $I(u)$ is bounded from above on H_k;*

$$\sup_{u\in H_k} I(u) < 0, \tag{4.1}$$

and from below on $H_k^\perp$; there exists $R_k > 0$ such that

$$\inf_{\substack{u\in H_k^\perp \\ |||u|||=R_k}} I(u) > 0 \tag{4.2}$$

and

$$\inf_{\substack{u\in H_k^\perp \\ |||u|||<R_k}} I(u) > -\infty. \tag{4.3}$$

Proof. For some constant $d \ge 0$, we have $G_r(s) \ge \frac{1}{2}\alpha s^2 + d$, where $G_r(s) = \int_0^s g_r(\sigma)d\sigma$. For $u \in H_k$,

$$(Lu, u) \le (\lambda_k(\lambda_k - c) - r)\int_\Omega u^2 dx = \frac{\lambda_k(\lambda_k - c) - \lambda_{k+1}(\lambda_{k+1} - c)}{2}\int_\Omega u^2,$$

$$\int_\Omega G_r(u) \ge \frac{\alpha}{2}\int_\Omega u^2 + d|\Omega|,$$

so that

$$I(u) \le \frac{1}{2}\cdot\frac{\lambda_k(\lambda_k - c) - \lambda_{k+1}(\lambda_{k+1} - c)}{2}\int_\Omega u^2 - \frac{\alpha}{2}\int_\Omega u^2 - d|\Omega| < 0,$$

since $\frac{\lambda_k(\lambda_k-c)-\lambda_{k+1}(\lambda_{k+1}-c)}{2} < \alpha$. Thus the functional I is bounded from above on H_k. Next we will prove that (4.2) and (4.3) hold. To get our claim (4.2), it is enough to prove that:

$$\lim_{\substack{u\in H^\perp \\ |||u|||\to+\infty}} I(u) = +\infty.$$

We have

$$\begin{aligned}
&\lim_{\substack{u\in H_k^{\perp}\\ |||u|||\to+\infty}} I(u)\\
\geq\ & \lim_{\substack{u\in H_k^{\perp}\\ |||u|||\to\infty}} \frac{1}{2}(1-\frac{r}{\lambda_{k+1}(\lambda_{k+1}-c)})|||u|||^2 - \lim_{\substack{u\in H_k^{\perp}\\ |||u|||\to+\ infty}} \int_\Omega G_r(u)dx\\
\geq\ & \lim_{\substack{u\in H_k^{\perp}\\ |||u|||\to+\infty}} \frac{1}{2}(1-\frac{r}{\lambda_{k+1}(\lambda_{k+1}-c)})|||u|||^2 - \lim_{\substack{u\in H_k^{\perp}\\ |||u|||\to+\infty}} \frac{1}{2}\beta\int_\Omega u^2-\bar{b}|\Omega|\\
\geq\ & \lim_{\substack{u\in H_k^{\perp}\\ |||u|||\to+\infty}} \frac{1}{2}(1-\frac{r}{\lambda_{k+1}(\lambda_{k+1}-c)}-\frac{\beta}{\lambda_{k+1}(\lambda_{k+1}-c)})|||u|||^2-\bar{b}|\Omega|\\
&\longrightarrow +\infty,
\end{aligned}$$

since there exists $\bar{b}\in R$ such that $G_r(u)<\frac{1}{2}\beta u^2+\bar{b}$, and

$$\beta<\frac{\lambda_{k+1}(\lambda_{k+1}-c)-\lambda_k(\lambda_k-c)}{2}.$$

Now we will prove (4.3). Since $\lambda_{k+m}(\lambda_{k+m}-c)<g'(0)<\lambda_{k+m+1}(\lambda_{k+m+1}-c)$ and $g'(t)\leq\gamma<\lambda_{k+m+1}(\lambda_{k+m+1}-c)$, there exists $\lambda_{k+m}(\lambda_{k+m}-c)<\bar{\gamma}<\lambda_{k+m+1}(\lambda_{k+m+1}-c)$ and $\bar{d}\geq 0$ such that $G(u)<\frac{\bar{\gamma}}{2}u^2+\bar{d}$. Thus

$$\begin{aligned}
\inf_{\substack{u\in H_k^{\perp}\\ |||u|||<R}} I(u) &= \inf_{\substack{u\in H_k^{\perp}\\ |||u|||<R}} \{\frac{1}{2}|||u|||-\int_\Omega G(u)\}\\
&> \inf_{\substack{u\in H_k^{\perp}\\ |||u|||<R}} \{\frac{1}{2}(1-\frac{\bar{\gamma}}{\lambda_{k+1}(\lambda_{k+1}-c)})|||u|||^2-\bar{d}|\Omega|\}\\
&> -\infty.
\end{aligned}$$

□

LEMMA 4.3. *Under the same assumptions of Theorem 1.1, there exists* $\rho_k>0$ *such that*

$$\sup_{\substack{u\in H_k\\ |||u|||=\rho_k}} I(u)<0.$$

Proof. Let $L_\infty: H\to H$ be the linear operator defined by

$$(L_\infty u,v)=(\Delta^2u+c\Delta u)v-g'(\infty)\int_\Omega uvdx,$$

where $\lambda_{i+1}(\lambda_{i+1}-c)<\lambda_k(\lambda_k-c)<g'(\infty)<\lambda_{k+1}(\lambda_{k+1}-c)$, $k>i+1$. Then L_∞ is an isomorphism. The spaces H_k, and $H_k^{\perp}$ are the negative space of L_∞ and the positive space of L_∞ respectively, and

$$H = H_k \oplus H_k^\perp.$$

Set $G_\infty(s) = G(s) - \frac{1}{2}g'(\infty)s^2$. Then

$$I(u) = \frac{1}{2}(L_\infty u, u) - \int_\Omega G_\infty(s)dx.$$

Thus, by Lemma 4.2, $\lim_{\substack{u \in H \\ u \to 0}} \frac{1}{|||u|||^2} \int_\Omega G_\infty(u)dx \geq 0$. Then

$$\begin{aligned} \lim_{\substack{u \in H_k \\ u \to 0}} \frac{I(u)}{|||u|||^2} &< \lim_{\substack{u \in H_k \\ u \to 0}} \frac{1}{2|||u|||^2}[\lambda_k(\lambda_k - c) - g'(\infty)] \int_\Omega u^2 \\ &- \lim_{\substack{u \in H_k \\ u \to 0}} \frac{1}{|||u|||^2} \int_\Omega G_\infty(u)dx < 0. \end{aligned}$$

thus there exists $\rho_k > 0$ such that

$$\sup_{\substack{u \in H_k \\ |||u|||=\rho_k}} < 0.$$

□

LEMMA 4.4. *Under the same assumptions of Theorem 1.1,*

$$\inf_{\substack{z \in H_k^\perp, \sigma \geq 0 \\ |||z-\sigma e_1|||=R_k}} I(z - \sigma e_1) \geq 0.$$

Proof. By Lemma 4.2, there exists $R_k > 0$ such that

$$\inf_{\substack{u \in H_k^\perp \\ |||u|||=R_k}} I(u) > 0.$$

To get our claim, it is enough to prove that

$$\lim_{\substack{z \in H_k^\perp, \sigma \geq 0, \\ |||z-\sigma e_1||| \to +\infty}} I(z - \sigma e_1) = +\infty. \tag{4.4}$$

To prove (4.4), we need to show that

$$\max_{\substack{z \in H_k^\perp \\ |||z|||=1}} \int z^2 = \max_{\substack{z \in H_k^\perp, \sigma \geq 0, \\ |||z-\sigma e_1|||=1}} \int (z - \sigma e_1)^2. \tag{4.5}$$

In fact, we have immediately $\max_{\substack{z \in H_k^\perp \\ |||z|||=1}} \int z^2 \leq \max_{\substack{z \in H_k^\perp, \sigma \geq 0 \\ |||z-\sigma e_1|||=1}} \int (z - \sigma e_1)^2$. Now we prove that $\max_{\substack{z \in H_k^\perp \\ |||z|||=1}} \int z^2 \geq \max_{\substack{z \in H_k^\perp, \sigma \geq 0 \\ |||z-\sigma e_1|||=1}} \int (z - \sigma e_1)^2$.

If $\sigma > 0$, then

$$2\int (z-\sigma e_1)v = \nu(z-\sigma e_1, v), \qquad \forall v \in H_1 \oplus H_k^{\perp}.$$

Taking $v = z - \sigma e_1$ we get $\nu = 2\int (z-\sigma e_1)^2$ and taking $v = e_1$ we also get

$$0 \le 2\int (z-\sigma e_1)e_1 = 2\int (z-\sigma e_1)^2(z-\sigma e_1, e_1) = -2\sigma\int (z-\sigma e_1)^2 < 0$$

which gives a contradiction. Then $z - \sigma e_1 = z \in H_k^{\perp}$ and so

$$\max_{\substack{z\in H_k^{\perp}\\ |||z-\sigma e_1|||=1}} \int (z-\sigma e_1)^2 = \max_{\substack{z\in H_k^{\perp}\\ |||z|||=1}} \int z^2.$$

Thus we proved (4.5). Now we prove (4.4). For some constant β, $b \ge 0$, we have $G_\infty(s) \ge \frac{1}{2}s^2 + b$, where $G_\infty(s) = \int_0^s g_\infty(\sigma)d\sigma$, $g_\infty(s) = g(s) - g'(\infty)s$. For $z \in H_k^{\perp}$ and $\sigma \ge 0$, by (4.5) we get

$$\begin{aligned}
& I(z-\sigma e_1)\\
\ge\ & \frac{1}{2}|||z-\sigma e_1|||^2 - \frac{1}{2}g'(\infty)\int_\Omega (z-\sigma e_1)^2 - \frac{1}{2}\beta\int_\Omega (z-\sigma e_1)^2 - b|\Omega|\\
=\ & \frac{1}{2}|||z-\sigma e_1|||^2(1-g'(\infty)\int \frac{(z-\sigma e_1)^2}{|||z-\sigma e_1|||^2} - \beta\int \frac{(z-\sigma e_1)^2}{|||z-\sigma e_1|||^2}) - b|\Omega|\\
\ge\ & \frac{1}{2}|||z-\sigma e_1|||^2(1-(g'(\infty)+\beta)\max_{z\in H_k^{\perp},\sigma\ge 0}\int \frac{(z-\sigma e_1)^2}{|||z-\sigma e_1|||^2}) - b|\Omega|\\
\ge\ & \frac{1}{2}|||z-\sigma e_1|||^2(1-(g'(\infty)+\beta)\max_{\substack{z\in H_k^{\perp}\\ |||z|||=1}}\int z^2) - b|\Omega| \longrightarrow \infty
\end{aligned}$$

as $|||z-\sigma e_1||| \to +\infty$. Thus we proved the lemma. □

From Lemma 4.3 and Lemma 4.4 we have

LEMMA 4.5. *Under the same assumptions of Theorem 1.1, there exists* $\rho_k > 0$ *such that*

$$\sup_{\substack{u\in H_k\\ |||u|||=\rho_k}} I(u) \le \inf_{z\in\Sigma(-e_1,H_k^{\perp})} I(z-\sigma e_1),$$

where $\Sigma(-e_1, H_k^{\perp}) = \{z \in H_k^{\perp} | |||z||| \le R_k\} \cup \{z-\sigma e_1 | z \in H_k^{\perp}, \sigma \ge 0, |||z-\sigma e_1||| = R_k\}$, *with* $R_k > \rho_k$.

LEMMA 4.6. *Let $G_0 : R \to R$ be a continuous function such that*

$$\inf_{s\in R} \frac{G_0(s)}{1+s^2} > -\infty, \qquad \lim_{s\to 0} \frac{G_0(s)}{s^2} \geq 0.$$

Then

$$\lim_{\substack{u\to 0\\ u\in H}} \frac{1}{|||u|||^2} \int_\Omega G_0(u)dx \geq 0.$$

Proof.

$$h(s) = \begin{cases} (\frac{G_o(s)}{s^2})^- & \text{if } s \neq 0, \\ 0 & \text{if } s = 0. \end{cases}$$

Then $h : R \to R$ is bounded, continuous, with $h(0) = 0$ and $G_0(s) \geq -h(s)s^2$. If (u_n) is a sequence in H with $u_n \to 0$, then up to a subsequence, $u_n \to 0$ a.e., and $v_n = \frac{u_n}{|||u_n|||}$ is strongly convergent in $L^2(\Omega)$. Since

$$\frac{1}{|||u_n|||^2} \int_\Omega G_0(u_n)dx \geq -\int_\Omega h(u_n)v_n^2 dx,$$

the claim follows. □

LEMMA 4.7. *Under the same assumptions of Theorem 1.1, there exists $\rho_{k+m} > 0$ such that*

$$\sup_{\substack{u\in H_{k+m}\\ |||u|||=\rho_{k+m}}} I(u) < \inf_{z\in\Sigma(e_{k+m}, H^\perp_{k+m})} I(z),$$

where $\Sigma(e_{k+m}, H^\perp_{k+m}) = \{w \in H^\perp_{k+m} | \, |||w||| \leq R_{k+m}\} \cup \{w + \sigma e_{k+m} | w \in H^\perp_{k+m}, \sigma \geq 0, |||w + \sigma e_{k+m}||| = R_{k+m}\}$ with $R_{k+m} > \rho_{k+m}$.

Proof.

$$\sup_{\substack{u\in H_{k+m}\\ |||u|||=\rho_{k+m}, \rho\to 0}} I(u) < 0. \tag{4.4}$$

From the assumptions of Theorem 1.1, $\lambda_{k+m}(\lambda_{k+m} - c) < g'(0) < \lambda_{k+m+1}(\lambda_{k+m+1} - c)$, $m \geq 1$. Let $L_0 : H \to H$ be the linear operator defined by

$$(L_0 u, v) = (\Delta^2 u + c\Delta u)v - g'(0) \int_\Omega uv dx.$$

Then L_0 is an isomorphism. The space H_{k+m}, $H^\perp_{k+m}$ are the negative space of L_0 and the positive space of L_0, respectively, and

$$H = H_{k+m} \oplus H^\perp_{k+m}.$$

Set $G_0(s) = G(s) - \frac{1}{2}g'(0)s^2$. Then

$$I(u) = \frac{1}{2}(L_0 u, u) - \int_\Omega G_0(u)dx.$$

Note that $\inf \frac{G_0(s)}{1+s^2} > -\infty$, $\lim_{s\to 0} \frac{G_0(s)}{s^2} \geq 0$. Thus by Lemma 4.1, $\lim_{\substack{u\to 0\\ u\in H}} \frac{1}{|||u|||^2} \int_\Omega G_0(u)dx \geq 0$. Then

$$\begin{aligned}\lim_{\substack{u\to 0\\ u\in H_{k+m}}} \frac{I(u)}{|||u|||^2} &< \lim_{\substack{u\to 0\\ u\in H_{k+m}}} \frac{1}{2|||u|||^2}[\lambda_{k+m}(\lambda_{k+m} - c) - g'(0)] \int_\Omega u^2 \\ &- \lim_{\substack{u\to 0\\ u\in H_{k+m}}} \frac{1}{|||u|||^2} \int_\Omega G_0(u)dx < 0.\end{aligned}$$

Thus there exists $\rho_{k+m} > 0$ such that $\sup_{\substack{u\in H_{k+m}\\ |||u|||=\rho_{k+m},\rho\to 0}} I(u) < 0$. By Lemma 4.2, $\inf_{\substack{u\in H_k^\perp\\ |||u|||=R_k}} I(u) > 0$. Thus we have

$$\sup_{\substack{u\in H_{k+m}\\ |||u|||=\rho_{k+m},\rho_{k+m}\to 0}} I(u) < \inf_{\substack{u\in H_k^\perp\\ |||u|||=R_k}} I(u)$$

with $R_k > \rho_{k+m}$. In other words, there exists

$$e_{k+m} \in Span\{\phi_{k+1}, \ldots, \phi_{k+m}\}$$

such that

$$\sup_{\substack{u\in H_{k+m}\\ |||u|||=\rho_{k+m},\rho_{k+m}\to 0}} I(u) < \inf_{\substack{u\in H_{k+m}^\perp \oplus e_{k+m}\\ e_{k+m}\in Span\{\phi_{k+1},\ldots,\phi_{k+n}\},|||u|||=R_{k+m}}} I(u).$$

□

PROOF OF THEOREM 1.1. AND THEOREM 1.2.

By Lemma 4.5, there exists $\rho_k > 0$ such that

$$\sup_{\substack{u\in H_k\\ |||u|||=\rho_k}} I(u) \leq \inf_{z\in\Sigma(-e_1,H_k^\perp)} I(z - \sigma e_1),$$

where $\Sigma(-e_1, H_k^\perp) = \{z \in H_k^\perp | |||z||| \leq R_k\} \cup \{z - \sigma e_1 | z \in H_k^\perp, \sigma \geq 0, |||z - \sigma e_1||| = R_k\}$, with $R_k > \rho_k$. By Lemma 4.7, there exists $\rho_{k+m} > 0$ such that

$$\sup_{\substack{u\in H_{k+m}\\ |||u|||=\rho_{k+m}}} I(u) < \inf_{z\in\Sigma(e_{k+m},H_{k+m}^\perp)} I(z),$$

where $\Sigma(e_{k+m}, H_{k+m}^\perp) = \{w \in H_{k+m}^\perp | |||w||| \leq R_{k+m}\} \cup \{w + \sigma e_{k+m} | w \in H_{k+m}^\perp, \sigma \geq 0, |||w + \sigma e_{k+m}||| = R_{k+m}\}$ with $R_{k+m} > \rho_{k+m}$ and $R_k >$

R_{k+m}. Thus by linking scale theorem 2.1., (1.1) has at least three solutions.

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