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# THREE POINT BOUNDARY VALUE PROBLEMS FOR THIRD ORDER FUZZY DIFFERENTIAL EQUATIONS

M.S.N. Murty* and G. Suresh Kumar**

Abstract. In this paper, we develop existence and uniqueness criteria to certain class of three point boundary value problems associated with third order nonlinear fuzzy differential equations, with the help of Green's functions and contraction mapping principle.

## 1. Introduction

The study of initial and boundary value problems for Fuzzy differential equations is an interesting area of current research.

Recently many authors [2]-[5] and [7] have studied initial and boundary value problems associated with first and second order Fuzzy differential equations on the metric space $(E^n, D)$ of normal fuzzy convex sets with the distance $D$ given by the supremum of the Hausdorff distance between the corresponding $\alpha$ - level sets.

In this direction existence and uniqueness theorem for initial value problems associated with first order fuzzy differential equation $y'(t) = f(t, y(t))$ was obtained by O.Kaleva[2] under usual assumptions of continuous and Lipschitz condition on $f$. Further J.J. Nieto [7] obtained a version of the Peano's existence theorem for fuzzy differential equations if $f$ is continuous and bounded.

Recently Lakshmikantham, Murty and Turner [4] obtained criteria for the existence and uniqueness solutions to two point boundary value problems associated with second order nonlinear fuzzy differential equations with the help of Greens functions and an application of Banach fixed point theorem.

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In this paper we present sufficient conditions for the existence and uniqueness of solutions of three point boundary value problems associated with a third order nonlinear fuzzy differential equations by using appropriate Greens function for the associated boundary value problems and by defining a contraction mapping that yields an interval over which a unique solution exists. Here we choose a suitable three point boundary value problem, where the signs of the Greens function and its partial derivatives over the different intervals can be established. This paper extended the results of V.Lakshmikantham etc. [4] developed for two point boundary value problems associated with second order fuzzy differential equations to three point boundary value problems for third order fuzzy differential equations.

## 2. Preliminaries

Let $P_k(R^n)$ denotes the family of all nonempty compact convex subsets of $R^n$. Define the addition and scalar multiplication in $P_k(R^n)$ as usual. Radstrom[9] states that $P_k(R^n)$ is a commutative semigroup under addition, which satisfies the cancellation law. Moreover, if $\alpha, \beta \in R$ and $A, B \in P_k(R^n)$ then

$$\alpha(A + B) = \alpha A + \alpha B, \quad \alpha(\beta A) = (\alpha\beta)A, \quad 1A = A$$

and if $\alpha, \beta \geq 0$, then $(\alpha+\beta)A = \alpha A+\beta A$. The distance between $A$ and $B$ is defined by the Hausdorff metric

$$d(A, B) = \inf\{\epsilon : A \subset N(B, \epsilon), B \subset N(A, \epsilon)\},$$

where

$$N(A, \epsilon) = \{x \in R^n : ||x - y|| < \epsilon, for\ \ some\ \ y \in A\}.$$

Let $I = [a, b] \subset R$ be a compact interval and denote

$$E^n = \{u : R^n \to [0, 1]/u\ \ satisfies\ \ (i) - (iv)\ \ below\}$$

where

(i) $u$ is normal, i.e there exists an $x_0 \in R^n$ such that $u(x_0) = 1$;

(ii) $u$ is fuzzy convex, i.e for $x, y \in R^n$ and $0 \leq \lambda \leq 1$,

$$u(\lambda x + (1 - \lambda)y) \geq \min[u(x), u(y)];$$

(iii) u is upper semicontinuous;

(iv)$[u]^0 = cl\{x \in R^n/u(x) > 0\}$ is compact.

For $0 < \alpha \leq 1$, the $\alpha$-level set is denoted and defined by $[u]^\alpha = \{x \in R^n / u(x) \geq \alpha\}$. Then from (i)-(iv) it follows that $[u]^\alpha \in P_k(R^n)$ for all $0 \leq \alpha \leq 1$.

Define $D : E^n \times E^n \to [0, \infty)$ by the equation

$$D(u, v) = \sup\{d([u]^\alpha, [v]^\alpha)/\alpha \in [0, 1]\},$$

where $d$ is the Hausdorff metric defined in $P_k(R^n)$. It is easy to show that $D$ is a metric in $E^n$ and using results of [[1], [8]], we see that $(E^n, D)$ is a complete metric space, but not locally compact. Moreover it has a linear structure in the sense that for all $u, v, w \in E^n, \lambda \in R$ we have that

$$D(u + w, v + w) = D(u, v), \quad D(\lambda u, \lambda v) = |\lambda| D(u, v).$$

We note that $(E^n, D)$ is not a vector space. But it can be imbedded isomorphically as a cone in a Banach space[9]. We define $\hat{0} \in E^n$ as $\hat{0}(x) = 1$ if $x = 0$ and $\hat{0}(x) = 0$ if $x \neq 0$. We state some theorems and lemmas which are useful for later discussion.

Theorem 2.1. *([2]) Let $F : I \to E^n$ be continuous. Then for all $t \in I$ the integral $G(t) = \int_a^t F(t)dt$ is differentiable and $G'(t) = F(t)$.*

Theorem 2.2. *([2]) Let $F : I \to E^n$ be continuously differentiable on $I$. Then*

$$D(F(b), F(a)) \leq (b - a) \sup\{D(F'(t), \hat{o})/t \in I\}.$$

Theorem 2.3. *(Ascoli's theorem) Let $X$ be a compact metric space and $Y$ any metric space. A subset $\Phi$ of the space $C(X, Y)$ of continuous mappings of $X$ into $Y$ is totally bounded in the metric of uniform convergence if and only if $\Phi$ is equicontinuous on $X$ and $\Phi(x) = \{\phi(x)/\phi \in \Phi\}$ is a totally bounded subset of $Y$ for each $x \in X$.*

## 3. First and second order fuzzy differential equations

In this section we briefly state the existence and uniqueness results for initial and boundary value problems associated with first and second order fuzzy differential equations.

Let $I = [a, b] \subset R$ and $f : I \times E^n \to E^n$ be continuous. A mapping $\phi : I \to E^n$ is a solution of the initial value problem

$$y' = f(t, y), \quad y(a) = y_0 \tag{1}$$

if and only if $\phi$ is a solution of the integral equation

$$y(t) = y_0 + \int_a^t f(s, y(s))ds \tag{2}$$

THEOREM 3.1. *([7]) Suppose that $f : I \times E^n \to E^n$ is continuous and bounded i.e there exists $r \geq 0$ such that*

$$D(f(t, y), \hat{o}) \leq r, \quad t \in I, \quad y \in E^n. \tag{3}$$

*Then the initial value problem (3.1) possesses at least one solution on the interval $I$.*

Consider the non-linear fuzzy differential equation of second order

$$y'' = f(t, y, y') \tag{4}$$

satisfying

$$y(a) = y_1 \quad y'(a) = m \tag{5}$$

where $f : I \times E^n \times E^n \to E^n$ is continuous.

If $\phi$ is a solution of (3.4) satisfying (3.5) if and only if $\phi$ is a solution of the integral equation

$$y(t) = y_1 + m(t - a) + \int_a^t (t - s)f(s, y(s), y'(s))ds \tag{6}$$

THEOREM 3.2. *([4]) Suppose that $f : I \times E^n \times E^n \to E^n$ is continuous and suppose there exists an $M > 0$ such that $D(f(t, y, y'), \hat{o}) \leq M$. Then the initial value problem (3.4) satisfying (3.5) possesses at least one solution on the interval $I$.*

THEOREM 3.3 (4). *Let $f \in C(I \times E^n \times E^n, E^n)$ and satisfy*

$$D[f(t, u, u'), f(t, v, v')] \leq KD(u, v) + LD(u', v')$$

*and assume that*

$$\alpha = K\frac{(b - a)^2}{8} + L\frac{(b - a)}{2} < 1.$$

*Then the two- point fuzzy boundary value problem*

$$y'' = f(t, y, y') \tag{7}$$

$$y(a) = y_1 \quad y(b) = y_2. \tag{8}$$

*has one and only one solution.*

## 4. Third order fuzzy differential equations

Let $J = [0, c]$ be a closed subinterval of $R$ and assume that $f : J \times E^n \times E^n \times E^n \to E^n$ is continuous. we consider the non-linear fuzzy differential equation of third order

$$y''' = f(t, y, y', y'') \tag{9}$$

satisfying three point boundary condition

$$y'(0) = y_1 \quad y(b) = y_2, \quad y''(c) = y_3. \tag{10}$$

Denote by $C^2(J, E^n)$ the set of all continuous twice differentiable mappings from $J$ to $E^n$. We define for any $\phi, \psi \in C^2(J, E^n)$ by

$$H(\phi, \psi) = K \max_{t \in J} D(\phi(t), \psi(t)) + L \max_{t \in J} D(\phi'(t), \psi'(t)) + M \max_{t \in J} D(\phi''(t), \psi''(t)).$$

Then $(C^2(J, E^n), H)$ is a complete metric space. For any $\phi \in C^2(J, E^n)$ define $F\phi \in C^2(J, E^n)$ by

$$[F\phi](t) = \int_0^c G(t, s) f(s, \phi(s), \phi'(s), \phi''(s)) ds, \ \ \forall \ t \in J.$$

Where $G(t, s)$ is the Green's function for the homogeneous boundary value problem. Hence $\phi \in C^2(J, E^n)$ is a solution of (4.1) satisfying (4.2) if and only if $\phi$ is a fixed point of $F$.

We know that, if $\phi \in C^2(J, E^n)$ is a solution of the initial value problem

$$y''' = f(t, y, y', y'') \tag{11}$$

$$y(0) = y_1, \quad y'(0) = m_1, \quad y''(0) = m_2 \tag{12}$$

if and only if $\phi$ is a solution of the integral equation

$$y(t) = y_1 + m_1 t + \frac{m_2 t^2}{2} + \int_0^t \frac{(t - s)^2}{2} f(s, y(s), y'(s), y''(s)) ds. \tag{13}$$

On the other hand, if we set $y'' = z$ then (4.3) and (4.4) becomes

$$z' = f(t, y, y', z) \tag{14}$$

$$y(0) = y_1, \quad y'(0) = m_1, \quad z(0) = m_2. \tag{15}$$

If $\psi$ is a solution of (4.6) satisfying (4.7), then $\psi$ is a solution of

$$z(t) = m_2 + \int_0^t f(s, y(s), y'(s), z(s))ds. \tag{16}$$

Now define for any $\phi, \psi \in C(J, E^n)$ by

$$H(\phi, \psi) = \sup\{D(\phi(t), \psi(t))/t \in J\}.$$

For any $\psi \in C(J, E^n)$ define $T\psi$ as

$$[T\psi](t) = m_2 + \int_0^t f(s, y(s), y'(s), z(s))ds. \tag{17}$$

LEMMA 4.1. *Suppose that there exists $N \geq 0$ such that*

$$D(f(t, y, y', y''), \hat{o}) \leq N, \ \forall \ t \in J, \quad y, y', y'' \in E^n. \tag{18}$$

*Then $T$ is compact. i.e $T$ transforms bounded sets into relatively compact sets.*

**Proof.** Let $B$ be a bounded set in $C(J, E^n)$. The set $TB = \{Ty/y \in B\}$ is totally bounded if and only if it is equicontinuous and for every $t \in J$ the set

$$[TB](t) = \{[Ty](t)/t \in J\}$$

is totally bounded subset of $E^n$. For any $t_0, t_1 \in J$ with $t_0 \leq t_1$ and $y \in B$ we have

$$\begin{aligned} D([Ty](t_0), [Ty](t_1)) &\leq |t_1 - t_0| \sup\{D(f(t, y(t), y'(t), Z(t)), \hat{0})/t \in J\} \\ &= |t_1 - t_0| \sup\{D(f(t, y(t), y'(t), y''(t)), \hat{0})/t \in J\} \\ &\leq |t_1 - t_0| N. \end{aligned}$$

Thus $TB$ is equicontinuous. Now for any fixed $t$ we have that

$$\begin{aligned} D([Ty](t), [Ty](t')) &\leq |t - t'| \sup\{D(f(t, y(t), y'(t), y''(t)), \hat{0})/t \in J\} \\ &\leq |t - t'| N \end{aligned}$$

for every $t' \in J, y \in B$.

Hence we see that $\{[Ty](t)/t \in J, y \in B\}$ is bounded in $E^n$. By Ascoli's theorem we conclude that $TB$ is a relatively compact subset of $C(J, E^n)$. This completes the proof of the lemma.

**Theorem 4.1** Suppose that $f : J \times E^n \times E^n \times E^n \to E^n$ is continuous and bounded, i.e., $f$ satisfies (4.10). Then the initial value problem (4.3) satisfies (4.4) possesses at least one solution on the interval $J$.

**Proof.** Consider the ball $B = \{\psi \in C(J, E^n)/H(\psi, \hat{0}) \leq N_1\}$ where $N_1 = cN$ in the metric space $(C(J, E^n), H)$. If $y \in TB, TB = \{T\phi/\phi \in B\}$, then $y = T\phi$ for some $\phi \in B$. Consider

$$D([T\phi](t),[T\phi](0)) = D([T\phi](t),\hat{0})$$
$$\leq |t-0| \sup\{D(f(t,\phi(t),\phi'(t),\phi''(t),\hat{0})/t \in J\}$$
$$\leq tN \leq cN = N_1.$$

Defining $\widetilde{0} : J \to E^n$, $\widetilde{0}(t) = \hat{0}$, $t \in J$, we have

$$H(T\phi, T\hat{0}) = \sup\{D([T\phi](t),[T\widetilde{0}](t))/t \in J\}$$
$$\leq N_1.$$

Therefore, $T\phi \in B$ and thus $TB \subset B$. By lemma 4.1, $T$ is compact and in consequence, it has a fixed point $\phi \in B$, which is a solution of (4.6) satisfying (4.7). Hence the theorem.

**Theorem 4.2** Let $f \in C[J \times E^n \times E^n \times E^n, E^n]$ and satisfy

(19)
$$D[f(t,u,u',u''), f(t,v,v',v'')] \leq KD(u,v) + LD(u',v') + MD(u'',v'')$$

and assume that

$$\beta = K\frac{b^2(3c-b)}{6} + L\frac{c^2}{2} + Mc < 1. \tag{20}$$

Then the three point fuzzy boundary value problem

$$y''' = f(t,y,y',y'') \tag{21}$$

$$y'(0) = y_1 \quad y(b) = y_2, \quad y''(c) = y_3. \tag{22}$$

has one and only one solution.

**Proof.** Clearly the homogeneous boundary value problem has only the trivial solution. Hence the three point boundary value problem (4.13) satisfying the boundary condition

$$y'(0) = 0 \quad y(b) = 0, \quad y''(c) = 0. \tag{23}$$

has a unique solution given by

$$y(t) = \int_0^c G(t,s) f(s,y(s),y'(s),y''(s))ds,$$

where

$$\underset{s\in[0,b]}{G(t,s)} = \begin{cases} \frac{s(2b-s)-t^2}{2} & if \quad t \leq s \\ s(b-t) & if \quad s < t \end{cases}$$

$$\underset{s\in[b,c]}{G(t,s)} = \begin{cases} \frac{b^2-t^2}{2} & if \quad t \leq s \\ \frac{b^2+s^2}{2} - ts. & if \quad s < t \end{cases}$$

The signs of the Green's function and its partial derivatives are given below over different intervals.

| $Function$ | $t \leq s \leq b,$ | $t \leq b < s,$ | $b < t < s,$ | $s < t \leq b,$ | $s \leq b < t,$ | $b < s < t$ |
|---|---|---|---|---|---|---|
| $G(t,s)$ | $nn$ | $nn$ | $-$ | $nn$ | $np$ | $-$ |
| $G_t(t,s)$ | $np$ | $np$ | $-$ | $np$ | $np$ | $-$ |
| $G_{tt}(t,s)$ | $-$ | $-$ | $-$ | 0 | 0 | 0 |

where $nn$:non-negative, $np$:non-positive.
It can be easily shown by elementary methods that

$$\max_{0\leq t\leq c}\int_0^c |G(t,s)|ds \leq \frac{b^2(3c-b)}{6}, \tag{24}$$

$$\max_{0\leq t\leq c}\int_0^c |G_t(t,s)|ds \leq \frac{c^2}{2}, \tag{25}$$

and

$$\max_{0\leq t\leq c}\int_0^c |G_{tt}(t,s)|ds \leq c. \tag{26}$$

Now we define $F : C^2(J,E^n) \to C^2(J,E^n)$ by

$$[Fu](t) = \int_0^c G(t,s)f(s,u(s),u'(s),u''(s))ds \tag{27}$$

for all $u \in C^2(J,E^n)$, $t \in J = [0,c]$.
Using the bounds on $G, G_t$ and $G_{tt}$ given by (4.16),(4.17) and (4.18), the definition of $Fu$ and from (4.11) we have

$$\begin{aligned} D([Fu](t),[Fv](t)) &\leq \int_0^c |G(t,s)|[KD(u(s),v(s)) + LD(u'(s),v'(s)) \\ &\qquad + MD(u''(s),v''(s))]\mathrm{ds} \\ &\leq H(u,v)\int_0^c |G(t,s)|ds \\ &\leq \frac{b^2(3c-b)}{6}H(u,v), \end{aligned} \tag{28}$$

$$D([Fu]'(t),[Fv]'(t)) \leq \int_0^c |G_t(t,s)|[KD(u(s),v(s)) + LD(u'(s),v'(s))$$

$$+MD(u''(s),v''(s))]\mathrm{ds}$$

$$\leq H(u,v)\int_0^c |G_t(t,s)|ds \leq \frac{c^2}{2}H(u,v), \tag{29}$$

and

$$D([Fu]''(t),[Fv]''(t)) \leq \int_0^c |G_{tt}(t,s)|[KD(u(s),v(s)) + LD(u'(s),v'(s))$$

$$+MD(u''(s),v''(s))]\mathrm{ds}$$

$$\leq H(u,v)\int_0^c |G_{tt}(t,s)|ds \leq cH(u,v). \tag{30}$$

Together with (4.20), (4.21) and (4.22) we have

$$H(Fu,Fv) \leq [K\frac{b^2(3c-b)}{6} + L\frac{c^2}{2} + Mc]H(u,v)$$
$$\leq \beta H(u,v).$$

From (4.12) $\beta = K\frac{b^2(3c-b)}{6} + L\frac{c^2}{2} + Mc < 1$. It follows that $F$ is a contraction mapping in the complete metric space $C^2((J,E^n),H)$. By contraction mapping theorem $F$ has a unique fixed point u, which is a unique solution of the boundary value problem (4.13) satisfying (4.15).

By applying the above procedure to the boundary value problem

$$y''' = f(t, y(t) + p(t), y'(t) + p'(t), y''(t) + p''(t))$$
$$y'(0) = 0 \quad y(b) = 0, \quad y''(c) = 0$$

where $p$ is a polynomial of second degree such that $p'(a) = y_1$, $p(b) = y_2$, $p''(c) = y_3$ a unique solution $y_1(t)$ is constructed. Let $y(t) = y_1(t)+p(t)$. Then it is easily seen that $y$ is a solution of the boundary value problem (4.13) and (4.14). Hence the theorem.

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Department of Applied Mathematics
Acarya Nagarjuna University Post Graduate Centre
NUZVID–521 201, Krishna (Dt.), Andra Pradesh
India
*E-mail*: drmsn2002@yahoo.com

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Department of Applied Mathematics
Acarya Nagarjuna University Post Graduate Centre
NUZVID–521 201, Krishna (Dt.), Andra Pradesh
India
*E-mail*: drmsn2002@yahoo.com