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COMPLETION OF QUASI–NORMED ALGEBRAS AND QUASI–NORMED MODULES

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Abstract. In this paper, the completion of a quasi-normed algebra and the completion of a quasi-normed module over a Banach algebra are constructed.

1. Introduction

It is well-known that the rational line $\mathbb{Q}$ is not complete but can be *enlarged* to the real line $\mathbb{R}$ which is complete. And this *completion* $\mathbb{R}$ of $\mathbb{Q}$ is such that $\mathbb{Q}$ is dense in $\mathbb{R}$. It is quite important that an arbitrary incomplete normed space can be *completed* in a similar fashion.

Banach spaces play an important role in many branches of mathematics and its applications ([2], [4]–[8], [10]).

We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1. ([3], [9]) Let X be a linear space. A *quasi-norm* is a real-valued function on X satisfying the following:

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
(2) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.
(3) There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a *quasi-normed space* if $\|\cdot\|$ is a quasi-norm on X.

A *quasi-Banach space* is a complete quasi-normed space.

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A quasi-norm $\|\cdot\|$ is called a *p-norm* $(0 < p \leq 1)$ if

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a *p-Banach space.*

DEFINITION 2. ([1]) Let $(X, \|\cdot\|)$ be a quasi-normed space. The quasi-normed space $(X, \|\cdot\|)$ is called a *quasi-normed algebra* if X is an algebra and there is a constant $C > 0$ such that $\|xy\| \leq C\|x\| \cdot \|y\|$ for all $x, y \in X$.

A *quasi-Banach algebra* is a complete quasi-normed algebra.

If the quasi-norm $\|\cdot\|$ is a *p-norm* then the quasi-Banach algebra is called a *p-Banach algebra.*

DEFINITION 3. Let $(A, |\cdot|)$ be a Banach algebra and X a module over A. A *quasi-norm* is a real-valued function on X satisfying the following:

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
(2) $\|ax\| = |a| \cdot \|x\|$ for all $a \in A$ and all $x \in X$.
(3) There is a constant $K \geq 1$ such that $\|x+y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a *quasi-normed module over A* if $\|\cdot\|$ is a quasi-norm on X.

A *quasi-Banach module over A* is a complete quasi-normed module over A.

A quasi-norm $\|\cdot\|$ is called a *p-norm* $(0 < p \leq 1)$ if

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach module over A is called a *p-Banach module over A.*

In Section 2, the completion of a quasi-normed algebra is constructed.

In Section 3, the completion of a quasi-normed module over a Banach algebra is constructed.

2. Completion of quasi-normed algebras

In this section, we construct a completion of a quasi-normed algebra.

DEFINITION 4. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be quasi-normed algebras.

(1) A mapping $L : X \to Y$ is said to be *isometric* or an *isometry* if for all $x, y \in X$

$$\|Lx - Ly\|_Y = \|x - y\|_X.$$

(2) The algebra X is said to be *isometric* with the algebra Y if there exists a bijective isometry of X onto Y. The algebras X and Y are called *isometric algebras.*

THEOREM 1. *Let $X = (X, \|\cdot\|_X)$ be a quasi-normed algebra. Assume that the quasi-norm $\|\cdot\|$ is a p-norm. Then there exist a quasi-Banach algebra $\widehat{X}$ and an isometry L from X onto a subalgebra Y of $\widehat{X}$ which is dense in $\widehat{X}$. The algebra $\widehat{X}$ is unique up to isometry.*

Proof. We divide the proof into four steps.

Step I. We construct a quasi-Banach algebra $(\widehat{X}, \|\cdot\|_{\widehat{X}})$.

Let $\{x_n\}$ and $\{x'_n\}$ be Cauchy sequences in X. Define $\{x_n\}$ to be equivalent to $\{x'_n\}$, written $\{x_n\} \sim \{x'_n\}$, if

$$lim_{n\to\infty} \|x_n - x'_n\|_X = 0. \tag{2.1}$$

Let $\widehat{X}$ be the set of all equivalence classes of Cauchy sequences. We write $\{x_n\} \in \widehat{x}$ to mean $\{x_n\}$ is a member of $\widehat{x}$ and a *representative* of the class $\widehat{x}$.

We must make $\widehat{X}$ into an algebra. To define on $\widehat{X}$ the three algebraic operations of an algebra, we consider any $\widehat{x}, \widehat{y} \in \widehat{X}$ and representatives $\{x_n\} \in \widehat{x}$ and $\{y_n\} \in \widehat{y}$. We set $z_n = x_n + y_n$. Then $\{z_n\}$ is Cauchy in X since

$$\|z_n - z_m\|_X = \|x_n + y_n - (x_m + y_m)\|_X \le K\|x_n - x_m\|_X + K\|y_n - y_m\|_X.$$

We define the sum $\widehat{z} = \widehat{x} + \widehat{y}$ of $\widehat{x}$ and $\widehat{y}$ to be the equivalence class for which $\{z_n\}$ is a representative, i.e., $\{z_n\} \in \widehat{z}$. This definition is independent of the particular choice of Cauchy sequences belonging to $\widehat{x}$ and $\widehat{y}$. In fact, the equality (2.1) shows that if $\{x_n\} \sim \{x'_n\}$ and $\{y_n\} \sim \{y'_n\}$, then $\{x_n+y_n\} \sim \{x'_n + y'_n\}$ because

$$\|x_n + y_n - (x'_n + y'_n)\|_X \le K\|x_n - x'_n\|_X + K\|y_n - y'_n\|_X.$$

Next, we consider any $\widehat{x}, \widehat{y} \in \widehat{X}$ and representatives $\{x_n\} \in \widehat{x}$ and $\{y_n\} \in \widehat{y}$. One can easily show that the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. We set $z_n = x_n \cdot y_n$. Then $\{z_n\}$ is Cauchy in X since

$$\begin{aligned}\|z_n - z_m\|_X = \|x_n \cdot y_n - x_m \cdot y_m\|_X &\le K\|(x_n - x_m)y_n\|_X + K\|x_m(y_n - y_m)\|_X \\ &\le KC\|x_n - x_m\|_X \cdot \|y_n\|_X + KC\|x_m\|_X \cdot \|y_n - y_m\|_X.\end{aligned}$$

We define $\widehat{z} = \widehat{x} \cdot \widehat{y}$ of $\widehat{x}$ and $\widehat{y}$ to be the equivalence class for which $\{z_n\}$ is a representative, i.e., $\{z_n\} \in \widehat{z}$. This definition is independent of the particular choice of Cauchy sequences belonging to $\widehat{x}$ and $\widehat{y}$. In fact, the equality (2.1) shows that if $\{x_n\} \sim \{x'_n\}$ and $\{y_n\} \sim \{y'_n\}$, then $\{x_n \cdot y_n\} \sim \{x'_n \cdot y'_n\}$ because

$$\begin{aligned}\|x_n \cdot y_n - x'_n \cdot y'_n)\|_X &\le K\|(x_n - x'_n)y_n\|_X + K\|x'_n(y_n - y'_n)\|_X \\ &\le KC\|x_n - x'_n\|_X \cdot \|y_n\|_X + KC\|x'_n\|_X \cdot \|y_n - y'_n\|_X.\end{aligned}$$

Similarly, we define the product $\alpha\widehat{x} \in \widehat{X}$ of a scalar α and $\widehat{x}$ to be the equivalence class for which $\{\alpha x_n\}$ is a representative. Again, this definition is independent of the particular choice of a representative of $\widehat{x}$. The zero element of $\widehat{X}$ is the equivalence class containing all Cauchy sequences which converge to zero. It is not difficult to see that those three algebraic operations have all the properties required by the definition, so that $\widehat{X}$ is an algebra.

We now set

$$\|\widehat{x} - \widehat{y}\|_{\widehat{X}} = lim_{n\to\infty}\|x_n - y_n\|_X, \tag{2.2}$$

where $\{x_n\} \in \widehat{x}$ and $\{y_n\} \in \widehat{y}$. We show that this limit exists. We have

$$\|x_n - y_n\|_X^p \le \|x_n - x_m\|_X^p + \|x_m - y_m\|_X^p + \|y_m - y_n\|_X^p.$$

So

$$\|x_n - y_n\|_X^p - \|x_m - y_m\|_X^p \le \|x_n - x_m\|_X^p + \|y_m - y_n\|_X^p,$$

and a similar inequality with m and n interchanged, i.e.,

$$\|x_m - y_m\|_X^p - \|x_n - y_n\|_X^p \le \|x_n - x_m\|_X^p + \|y_m - y_n\|_X^p.$$

Hence

$$\big|\|x_n - y_n\|_X^p - \|x_m - y_m\|_X^p\big| \le \|x_n - x_m\|_X^p + \|y_m - y_n\|_X^p. \tag{2.3}$$

Since $\{x_n\}$ and $\{y_n\}$ are Cauchy, we can make the right side as small as we please. This implies that the limit in (2.2) exists since $\mathbb{R}$ is complete.

We must show that the limit in (2.2) is independent of the particular choice of representatives. If $\{x_n\} \sim \{x'_n\}$ and $\{y_n\} \sim \{y'_n\}$, then by (2.1)

$$\big|\|x_n - y_n\|_X^p - \|x'_n - y'_n\|_X^p\big| \le \|x_n - x'_n\|_X^p + \|y_n - y'_n\|_X^p,$$

which tends to zero as $n \to \infty$. This implies the assertion

$$\lim_{n\to\infty} \|x_n - y_n\|_X = \lim_{n\to\infty} \|x'_n - y'_n\|_X.$$

Now we prove that $\|\cdot\|_{\widehat{X}}$ in (2.2) is a quasi-norm on $\widehat{X}$.

Obviously, $\|\cdot\|_{\widehat{X}}$ satisfies Definition 1 (1) and (2). Furthermore, since $\|\cdot\|_X$ is a quasi-norm on X, there is a constant $K \ge 1$ such that

$$\|x + y\|_X \le K(\|x\|_X + \|y\|_X)$$

for all $x, y \in X$. Thus

$$\|\widehat{x} + \widehat{y}\|_{\widehat{X}} \le K(\|\widehat{x}\|_{\widehat{X}} + \|\widehat{y}\|_{\widehat{X}})$$

for all $\widehat{x}, \widehat{y} \in \widehat{X}$. So $\|\cdot\|_{\widehat{X}}$ is a quasi-norm on $\widehat{X}$.

Note that the inequality (2.2) implies that the quasi-norm $\|\cdot\|_{\widehat{X}}$ is equivalent to a p-norm.

Step II. We construct an isometry $L : X \to Y \subset \widehat{X}$.

With each $b \in X$ we associate the class $\widehat{b} \in \widehat{X}$ which contains the constant Cauchy sequence $(b, b \cdots)$. This defines a mapping $L : X \to Y$ onto the subalgebra $Y = L(X) \subset \widehat{X}$. The mapping L is given by $b \mapsto \widehat{b} = Lb$, where $(b, b, \cdots) \in \widehat{b}$. We see that L is an isometry since (2.2) becomes simply

$$\|\widehat{b} - \widehat{c}\|_{\widehat{X}} = \|b - c\|_X.$$

Here $\widehat{c}$ is the class of $\{y_n\}$ where $y_n = c$ for all $n \in \mathbb{N}$. Any isometry is injective, and $L : X \to Y$ is surjective since $L(X) = Y$. Hence Y and X are isometric.

From the definition it follows that on Y the operations of algebra induced from $\widehat{X}$ agree with those induced from X by means of L.

We show that Y is dense in $\widehat{X}$. We consider any $\widehat{x} \in \widehat{X}$. Let $\{x_n\} \in \widehat{x}$. For every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$\|x_n - x_N\|_X < \frac{\epsilon}{2}$$

for all $n > N$. Let $(x_N, x_N, \cdots) \in \widehat{x_N}$. Then $\widehat{x_N} \in Y$. By (2.2),

$$\|\widehat{x} - \widehat{x_N}\|_{\widehat{X}} = \lim_{n \to \infty} \|x_n - x_N\|_X \le \frac{\epsilon}{2} < \epsilon.$$

This shows that every ϵ-neighborhood of the arbitrary $\widehat{x} \in \widehat{X}$ contains an element of Y. Hence Y is dense in $\widehat{X}$.

Step III. We prove the completeness of $\widehat{X}$.

Let $\{\widehat{x_n}\}$ be any Cauchy sequence in $\widehat{X}$. Since Y is dense in $\widehat{X}$, for every $\widehat{x_n}$ there is a $\widehat{z_n} \in Y$ such that

$$\|\widehat{x_n} - \widehat{z_n}\|_{\widehat{X}} < \frac{1}{n}. \tag{2.4}$$

Hence by Definition 1 (3),

$$\begin{aligned}\|\widehat{z_m} - \widehat{z_n}\|_{\widehat{X}} &\le K\|\widehat{z_m} - \widehat{x_m}\|_{\widehat{X}} + K\|\widehat{x_m} - \widehat{x_n}\|_{\widehat{X}} + K\|\widehat{x_m} - \widehat{z_n}\|_{\widehat{X}} \\ &< \frac{K}{m} + K\|\widehat{x_m} - \widehat{x_n}\|_{\widehat{X}} + \frac{K}{n}\end{aligned}$$

and this is less than any given $\epsilon > 0$ for sufficiently large m and n because $\{\widehat{x_n}\}$ is a Cauchy sequence. Hence $\{\widehat{z_n}\}$ is a Cauchy sequence. Since L :

$X \to Y$ is isometric and $\widehat{z_n} \in Y$, the sequence $\{z_n\}$, where $z_n = L^{-1}(\widehat{z_n})$, is a Cauchy sequence in X. Let $\widehat{x} \in \widehat{X}$ be the class to which $\{z_n\}$ belongs. We show that $\widehat{x}$ is the limit of $\{\widehat{x_n}\}$. By Definition 1 (3) and (2.4),

$$\begin{aligned}\|\widehat{x_n} - \widehat{x}\|_{\widehat{X}} &\leq K\|\widehat{x_n} - \widehat{z_n}\|_{\widehat{X}} + K\|\widehat{z_n} - \widehat{x}\|_{\widehat{X}} \\ &< \frac{K}{n} + K\|\widehat{z_n} - \widehat{x}\|_{\widehat{X}}.\end{aligned} \tag{2.5}$$

Since $\{z_n\} \in \widehat{x}$ and $\widehat{z_m} \in Y$, so that $(z_m, z_m, z_m, \cdots) \in \widehat{z_m}$, the inequality (2.5) becomes

$$\|\widehat{x_n} - \widehat{x}\|_{\widehat{X}} < \frac{K}{n} + K \lim_{m\to\infty} \|z_n - z_m\|_X$$

and the right side is smaller than any given $\epsilon > 0$ for sufficiently large n. Hence the arbitrary Cauchy sequence $\{\widehat{x_n}\}$ in $\widehat{X}$ has the limit $\widehat{x} \in \widehat{X}$, and $\widehat{X}$ is complete.

Step IV. We show the uniqueness of $\widehat{X}$ up to isometry.

If $(\widetilde{X}, \|\cdot\|_{\widetilde{X}})$ is another complete metric algebra with a subalgebra Z dense in $\widetilde{X}$ and isometric with X, then for any $\widetilde{x}, \widetilde{y} \in \widetilde{X}$ we have sequences $\{\widetilde{x_n}\}, \{\widetilde{y_n}\}$ in Z such that $\widetilde{x_n} \to \widetilde{x}$ and $\widetilde{y_n} \to \widetilde{y}$. So

$$\|\widetilde{x} - \widetilde{y}\|_{\widetilde{X}} = \lim_{n\to\infty} \|\widetilde{x_n} - \widetilde{y_n}\|_{\widetilde{X}}$$

follows from

$$\left|\|\widetilde{x} - \widetilde{y}\|^p_{\widetilde{X}} - \|\widetilde{x_n} - \widetilde{y_n}\|^p_{\widetilde{X}}\right| \leq \|\widetilde{x} - \widetilde{x_n}\|^p_{\widetilde{X}} + \|\widetilde{y} - \widetilde{y_n}\|^p_{\widetilde{X}} \to 0.$$

Here the inequality is similar to (2.3). Since Z is isometric with $Y \subset \widehat{X}$ and $\overline{Y} = \widehat{X}$, the norms on $\widetilde{X}$ and $\widehat{X}$ must be the same. Hence $\widetilde{X}$ and $\widehat{X}$ are isometric. □

3. Completion of quasi-normed modules over a Banach algebra

In this section, we construct a completion of a quasi-normed module over a Banach algebra.

Definition 5. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be quasi-normed modules over a Banach algebra A.

(1) A mapping $L : X \to Y$ is said to be *isometric* or an *isometry* if for all $x, y \in X$

$$\|Lx - Ly\|_Y = \|x - y\|_X.$$

(2) The A-module X is said to be *isometric* with the A-module Y if there exists a bijective isometry of X onto Y. The A-modules X and Y are called *isometric A-modules.*

Theorem 2. *Let $X = (X, \|\cdot\|_X)$ be a quasi-normed module over a Banach algebra A. Assume that the quasi-norm $\|\cdot\|$ is a p-norm. Then there exist a quasi-Banach module $\widehat{X}$ over A and an isometry L from X onto a submodule Y of $\widehat{X}$ which is dense in $\widehat{X}$. The quasi-Banach A-module $\widehat{X}$ is unique up to isometry.*

Proof. We construct a quasi-Banach A-module $(\widehat{X}, \|\cdot\|_{\widehat{X}})$.

Let $\{x_n\}$ and $\{x'_n\}$ be Cauchy sequences in X. Define $\{x_n\}$ to be equivalent to $\{x'_n\}$, written $\{x_n\} \sim \{x'_n\}$, if

$$lim_{n\to\infty}\|x_n - x'_n\|_X = 0. \tag{3.1}$$

Let $\widehat{X}$ be the set of all equivalence classes of Cauchy sequences. We write $\{x_n\} \in \widehat{x}$ to mean $\{x_n\}$ is a member of $\widehat{x}$ and a *representative* of the class $\widehat{x}$.

We must make $\widehat{X}$ into an A-module. To define on $\widehat{X}$ the two algebraic operations of a module, we consider any $\widehat{x}, \widehat{y} \in \widehat{X}$ and representatives $\{x_n\} \in \widehat{x}$ and $\{y_n\} \in \widehat{y}$. We set $z_n = x_n + y_n$. Then $\{z_n\}$ is Cauchy in X since

$$\|z_n - z_m\|_X = \|x_n + y_n - (x_m + y_m)\|_X \le K\|x_n - x_m\|_X + K\|y_n - y_m\|_X.$$

We define the sum $\widehat{z} = \widehat{x} + \widehat{y}$ of $\widehat{x}$ and $\widehat{y}$ to be the equivalence class for which $\{z_n\}$ is a representative, i.e., $\{z_n\} \in \widehat{z}$. This definition is independent of the particular choice of Cauchy sequences belonging to $\widehat{x}$ and $\widehat{y}$.

Similarly, we define the product $a\widehat{x} \in \widehat{X}$ of $a \in A$ and $\widehat{x}$ to be the equivalence class for which $\{ax_n\}$ is a representative. Again, this definition is independent of the particular choice of a representative of $\widehat{x}$. The zero element of $\widehat{X}$ is the equivalence class containing all Cauchy sequences which converge to zero. It is not difficult to see that those two algebraic operations have all the properties required by the definition, so that $\widehat{X}$ is an A-module.

We now set

$$\|\widehat{x} - \widehat{y}\|_{\widehat{X}} = lim_{n\to\infty}\|x_n - y_n\|_X, \tag{3.2}$$

where $\{x_n\} \in \widehat{x}$ and $\{y_n\} \in \widehat{y}$. By the same method as in the proof of Theorem 1, one can show that $\|\cdot\|_{\widehat{X}}$ in (3.2) is a quasi-norm on $\widehat{X}$.

Next, we construct an isometry $L : X \to Y \subset \widehat{X}$.

With each $b \in X$ we associate the class $\widehat{b} \in \widehat{X}$ which contains the constant Cauchy sequence $(b, b \cdots)$. This defines a mapping $L : X \to Y$ onto the submodule $Y = L(X) \subset \widehat{X}$. The mapping L is given by $b \mapsto \widehat{b} = Lb$, where $(b, b, \cdots) \in \widehat{b}$. We see that L is an isometry since (3.2) becomes simply

$$\|\widehat{b} - \widehat{c}\|_{\widehat{X}} = \|b - c\|_X.$$

Here $\widehat{c}$ is the class of $\{y_n\}$ where $y_n = c$ for all $n \in \mathbb{N}$. Any isometry is injective, and $L : X \to Y$ is surjective since $L(X) = Y$. Hence Y and X are isometric.

From the definition it follows that on Y the operations of module induced from $\widehat{X}$ agree with those induced from X by means of L.

The rest of the proof is similar to the proof of Theorem 1. □

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