

## ON THE ORDER AND RATE OF CONVERGENCE FOR PSEUDO-SECANT-NEWTON'S METHOD LOCATING A SIMPLE REAL ZERO

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ABSTRACT. By combining the classical Newton's method with the pseudo-secant method, pseudo-secant-Newton's method is constructed and its order and rate of convergence are investigated. Given a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  that has a simple real zero  $\alpha$  and is sufficiently smooth in a small neighborhood of  $\alpha$ , the convergence behavior is analyzed near  $\alpha$  for pseudo-secant-Newton's method. The order of convergence is shown to be cubic and the rate of convergence is proven to be  $\left(\frac{f''(\alpha)}{2f'(\alpha)}\right)^2$ . Numerical experiments show the validity of the theory presented here and are confirmed via high-precision programming in Mathematica.

### 1.. Introduction and Preliminaries

The aim of this paper is to establish the order and rate of convergence for a variant of Newton's method called pseudo-secant-Newton's method. Although an analysis for the convergence behavior of this method was done by Kasturiarachi [5], a slightly different approach will be presented here under for a sufficiently smooth function  $f$  whose typical zero is to be sought. Indeed, the rate of convergence for this method derived by Kasturiarachi has turned out to be incorrect. This incorrecedness motivates the current analysis which will provide us with a completely correct expression for the rate of convergence of the pseudo-secant-Newton's method. Furthermore, various numerical examples to be shown with the high-precision computability of Mathematica [12] will doubtlessly strengthen the validity of the current expression.

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Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  have a simple real zero  $\alpha$  and be sufficiently smooth in a small neighborhood of  $\alpha$ . Given  $x_0 \in \mathbb{R}$ , for  $n \in \mathbb{N} \cup \{0\}$  Pseudo-secant-Newton's method is iteratively defined by

$$x_{n+1} = x_n - \frac{f(x_n)^2}{\{f(x_n) - f(\bar{x}_n)\} f'(x_n)}, \quad (1.1)$$

where

$$\bar{x}_n = x_n - \frac{f(x_n)}{f'(x_n)}. \quad (1.2)$$

Combining (1.1) with (1.2) immediately leads us to the iterative method below:

$$x_{n+1} = x_n - h \cdot \frac{f(x_n)}{f(x_n) - f(x_n - h)}, \quad (1.3)$$

where  $f = f(x_n)$ ,  $f' = f'(x_n)$ ,  $h = f(x_n)/f'(x_n)$  are used for brevity and the symbol  $'$  denotes the derivative with respect to  $x$ . Since  $f(x_n)$  is approximately equal to 0 in a neighborhood of  $\alpha$ , then the value of  $\lim_{n \rightarrow \infty} h = 0$  is assumed.

We expand  $f(x_n - h)$  by Talyor's series [1,9] about  $x = x_n$ , and take the first several terms up to the third-degree in  $h$  to obtain

$$\begin{aligned} f(x_n) - f(x_n - h) &= hf'(x_n) - \frac{h^2}{2}f''(x_n) + \frac{h^3}{6}f'''(x_n) + O(h^4) \\ &= hf'(x_n)(1 - u), \end{aligned} \quad (1.4)$$

where  $O(h^4)$  is Landau notation for the bounded remainder term of the corresponding Taylor's series when divided by  $h^4$ , and

$$u = \frac{hf''(x_n)}{2f'(x_n)} - \frac{h^2f'''(x_n)}{6f'(x_n)} + O(h^3). \quad (1.5)$$

Since  $f(x_n) \approx 0$  near  $\alpha$ , we find that  $h$  is small and  $|u| < 1$ . Therefore, the second term of the right side of (1.3) becomes, after expanding by Taylor series about  $u = 0$ ,

$$\begin{aligned} h \cdot \frac{f(x_n)}{f(x_n) - f(x_n - h)} &= h \cdot (1 + u + u^2 + u^3 + \cdots) \\ &= h \left\{ 1 + \frac{h^2}{2} \cdot \frac{f''}{f'} - \frac{h^2}{6} \cdot \frac{f'''}{f'} + O(h^3) \right\} \\ &\quad + h^2 \left( \frac{h}{2} \cdot \frac{f''}{f'} - \frac{h^2}{6} \cdot \frac{f'''}{f'} + O(h^3) \right)^2 + hO(h^7) \\ &= h \left\{ 1 + \frac{h}{2} \cdot \frac{f''}{f'} - \frac{h^2}{6} \cdot \frac{f'''}{f'} + \frac{h^2}{4} \frac{f''^2}{f'^2} \right\} + O(h^4). \end{aligned} \quad (1.6)$$

Putting  $\alpha = x_n + \alpha - x_n$  and expanding  $f(\alpha)$  by Taylor series about  $x_n$ , we obtain

$$0 = f(\alpha) = f(\alpha - x_n + x_n) = f(x_n) + (\alpha - x_n)f'(x_n) + \frac{(\alpha - x_n)^2}{2!}f''(x_n) + \frac{(\alpha - x_n)^3}{3!}f'''(c),$$

where  $c \in (a, b)$ ,  $a = \min(\alpha, x_n)$ ,  $b = \max(\alpha, x_n)$  and  $\lim_{n \rightarrow \infty} c = \alpha$ .

Multiplying by  $1/f'$  both sides of the above equation with the aid of (1.6) leads us to the following equations.

$$-h = (\alpha - x_n)\left[1 + \frac{(\alpha - x_n)}{2} \frac{f''}{f'} + \frac{(\alpha - x_n)^2}{6} \frac{f'''}{f'}\right]. \quad (1.7)$$

$$h^2 = (\alpha - x_n)^2\left[1 + (\alpha - x_n) \frac{f''}{f'} + \frac{(\alpha - x_n)^2}{4} \frac{f''^2}{f'^2} + O(\alpha - x_n^3)\right]. \quad (1.8)$$

Subtracting  $\alpha_n$  from both sides of (1.3) and using (1.6), we obtain the equation below:

$$x_{n+1} - \alpha = x_n - \alpha - h \left\{ 1 + \frac{h}{2} \cdot \frac{f''}{f'} - \frac{h^2}{6} \cdot \frac{f'''}{f'} + \frac{h^2}{4} \frac{f''^2}{f'^2} \right\} + O(h^4).$$

Substituting (1.7) into the first-order term in  $h$  on the the right side of the above equation immediately yields the equation below.

$$\begin{aligned} x_{n+1} - \alpha &= \frac{(\alpha - x_n)^2}{2} \frac{f''}{f'} + \frac{(\alpha - x_n)^3}{6} \frac{f'''}{f'} \\ &\quad - h^2 \left( \frac{f''}{2f'} - \frac{h}{6} \cdot \frac{f'''}{f'} + \frac{h}{4} \frac{f''^2}{f'^2} \right) + O(h^4) \\ &= \frac{(\alpha - x_n)^2}{2} \frac{f''}{f'} + \frac{(\alpha - x_n)^3}{6} \frac{f'''}{f'} \\ &\quad - (\alpha - x_n)^2 \left( 1 + (\alpha - x_n) \frac{f''}{f'} + O(\alpha - x_n^2) \right) \left( \frac{f''}{2f'} + dh \right) \end{aligned} \quad (1.9)$$

where  $d = \frac{f''^2}{4f'^2} - \frac{f'''}{6f'}$ . Further calculation shows that

$$x_{n+1} - \alpha = (\alpha - x_n)^3 \left\{ \frac{f'''(c)}{6f'} + \frac{f''^2}{4f'^2} - \frac{f'''(x_n)}{6f'} - \frac{f''^2}{f'^2} \right\} + O(\alpha - x_n^4).$$

As a result, we obtain the following equation

$$\frac{(x_{n+1} - \alpha)}{(x_{n+1} - \alpha)^3} = \frac{f''^2}{4f'^2} + \frac{f'''(x_n)}{6f'} - \frac{f'''(c)}{6f'} + O(x_n - \alpha).$$

By taking absolute values of both sides of the above equation and passing to the limit as  $n$  approaches infinity, we find that  $x_n \rightarrow \alpha$  and  $c \rightarrow \alpha$ . Hence the above equation now reduces to

$$\eta = \lim_{n \rightarrow \infty} \left| \frac{(x_{n+1} - \alpha)}{(x_{n+1} - \alpha)^3} \right| = \frac{f''^2(\alpha)}{4f'^2(\alpha)}, \quad (1.10)$$

which is the rate of convergence (asymptotic error constant) [2,3,10] with third-order convergence. Other numerical methods of order three can be found in [4,7,8,11]. Kasturiarachi [5] showed a different rate of convergence

$$\eta = \frac{|3 f''^2(\alpha) - 2f^{(3)}(\alpha)f'(\alpha)|}{6f'^2(\alpha)}$$

which seems incorrect due to the faulty manipulation of the Taylor series(1.7). The analysis discussed so far finally gives us the following theorem.

**THEOREM 1.1.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  have a simple real zero  $\alpha$  and be sufficiently smooth in a small neighborhood of  $\alpha$ . Then pseudo-secant-Newton's method converges with order three and its rate of convergence is found to be  $\eta = |c^2|/4$ , where  $c = f''(\alpha)/f'(\alpha)$ .

## 2.. Algorithm, numerical results and discussions

In this section, we construct a zero-finding algorithm with the support of symbolic and computational ability of *Mathematica* on the basis of the analysis shown in Section 1.

**CLAIM 2.2.** Algorithm 2.1 (Zero-finding algorithm)

*Step 1.* For  $k \in \mathbb{N} \cup \{0\}$ , construct the iteration scheme (1.3) with the given function  $f$  having a simple zero  $\alpha$ , according to the description in Section 1.

*Step 2.* Set the minimum number of precision digits. With exact zero  $\alpha$  or most accurate zero, supply the theoretical asymptotic error constant  $\eta$ . Set the error range  $\epsilon$ , the maximum iteration number  $n_{max}$  and the initial value  $x_0$ . Compute  $f(x_0)$  and  $|x_0 - \alpha|$ .

*Step 3.* Compute  $x_{n+1}$  in (1.3) for  $0 \leq n \leq n_{max}$  and display the computed values of  $n$ ,  $x_n$ ,  $f(x_n)$ ,  $|x_n - \alpha|$ ,  $|e_{n+1}/e_n^3|$  and  $\eta$ .

As a numerical example for the convergence of pseudo-secant-Newton's method, we first illustrate the order of convergence and asymptotic error

constant with a function

$$f(x) = (x^2 + 1) \cos(\pi x/8)$$

having a simple real zero  $\alpha = -4$ . The symbolic computation of  $f'(x)$  has been easily done with the aid of Mathematica. Table 1 lists the numerical results for approximated zeros of  $f(x)$  computed with Mathematica programming. To obtain sufficient accuracy, the minimum number of precision digits was chosen as 250 by assigning  $\$MinPrecision=250$  in Mathematica. The error bound  $\epsilon$  for  $|x_n - \alpha| < \epsilon$  was chosen as  $0.5 \times 10^{-235}$  for the current experiment. As can be seen in Table 1, the order of convergence has been confirmed to be cubic.

TABLE 1. Convergence of pseudo-secant-Newton method for  $f(x) = (x^2 + 1) \cos(\pi x/8)$

$n$	$x_n$	$f(x_n)$	$ x_n - \alpha $	$e_{n+1}/e_n^3$	$\eta$
0	-3.200000000000000	3.47335	0.800000		
1	-3.71842121657687	1.63613	0.281579	0.5499585614	
2	-3.99302114596102	0.0464371	0.00697885	0.3125966580	
3	-3.99999992409404	$5.06739 \times 10^{-7}$	$7.59060 \times 10^{-8}$	0.2233178953	0.2214532872
4	-4.000000000000009	$6.46575 \times 10^{-22}$	$9.68523 \times 10^{-23}$	0.2214533074	
5	-4.000000000000000	$1.34314 \times 10^{-66}$	$2.01192 \times 10^{-67}$	0.2214532872	
6	-4.000000000000000	$-1.20400 \times 10^{-200}$	$1.80350 \times 10^{-201}$	0.2214532872	
7	-4.000000000000000	$-4.85718 \times 10^{-260}$	$8.12987 \times 10^{-261}$		

As a second numerical example to confirm the convergence, we take

$$f(x) = z^{10} - 3z^3 e^{\cos x} - 1$$

with a simple real zero

$\alpha = 1.2454283753596838267131847481004361733761068230618687830979702334524850709991$   
 418357166929374480675215064276250438067891165701483628864187537195082756847310  
 407419180946411805425271578504085209021242807458038319574517728156996806134750  
 040589969392495511043028765346160610249940310166439070367768125915250967547643  
 327270283457200682630058710094332198956208601144085310955944071401694435332592  
 961742857525011284992107869083440420262665711921984261027063123895887529076764,

which is accurate up to 250 significant decimal digits. Table 2 also shows a good accordance with the theoretical ideas presented in this paper. The computed asymptotic error constants were found in good agreement with the theoretical asymptotic error constants  $\eta$  up to 10 significant digits. The computed root was rounded to be accurate up to the 235 significant digits. The limited space allows us to list it only up

TABLE 2. Convergence of pseudo-secant-Newton method  
for  $f(x) = z^{10} - 3z^3 e^{\cos x} - 1$

$n$	$x_n$	$f(x_n)$	$ x_n - \alpha $	$e_{n+1}/e_n^3$	$\eta$
0	1.10000000000000	-4.69109	0.145428		
1	1.13910928707585	-4.05959	0.106319	34.56716689	
2	1.00008863454853	-2.04922	0.0403693	33.59056353	
3	1.24383857765155	-0.0954153	0.00158980	24.16511149	
4	1.24542829965909	$-4.57464 \times 10^{-6}$	$7.57006 \times 10^{-8}$	18.83968769	18.64595504
5	1.24542837535968	$-4.88810 \times 10^{-19}$	$8.08877 \times 10^{-21}$	18.64596421	
6	1.24542837535968	$-5.96335 \times 10^{-58}$	$9.86808 \times 10^{-60}$	18.64595504	
7	1.24542837535968	$-1.08278 \times 10^{-174}$	$1.79177 \times 10^{-176}$	18.64595504	
8	1.24542837535968	$-3.82104 \times 10^{-259}$	$5.62073 \times 10^{-261}$		

to 15 significant digits. Additional numerical experiments clearly ensure cubic convergence of pseudo-secant-Newton's method.

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