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FREE ACTIONS OF FINITE GROUPS ON THE 3–DIMENSIONAL NILMANIFOLD FOR TYPE 1

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ABSTRACT. We study free actions of finite groups on the 3-dimensional nilmanifold for Type 1 and classify all such group actions, up to topological conjugacy. This work supplies missing one in [1, Theorem 3.11.].

1. Introduction

Infra-nilmanifolds are determined uniquely by their fundamental groups, called almost Bieberbach groups. It is known ([3; Proposition 6.1.]) that there are 15 classes of distinct closed 3-dimensional manifolds M with a Nil-geometry up to Seifert local invariant.

The general question of classifying finite group actions on a closed 3-manifold is very hard. However, the actions on a 3-dimensional nilmanifold can be understood easily by the works of Bieberbach, L. Auslander and Waldhausen([5, 6, 9]). Free actions of finite, cyclic and abelian groups on the 3-torus were studied in [4], [7] and [8], respectively. It is interesting that if a finite group acts freely on the 3-dimensional nilmanifold with the first homology $\mathbb{Z}^2$, then it is cyclic [2]. Free actions of finite abelian groups on the 3-dimensional nilmanifold with the first homology $\mathbb{Z}^2 \oplus \mathbb{Z}_p$ were classified in [1].

Let $\mathcal{H}$ be the 3–dimensional Heisenberg group; i.e. $\mathcal{H}$ consists of all 3×3 real upper triangular matrices with diagonal entries 1. That is,

$$\mathcal{H} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \;:\; x, y, z \in \mathbb{R} \right\}.$$

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Thus $\mathcal{H}$ is a simply connected, 2-step nilpotent Lie group.

For each integer $p > 0$, let

$$\Gamma_p = \left\{ \begin{bmatrix} 1 & l & \frac{n}{p} \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix} \middle| \; l, m, n \in \mathbb{Z} \right\}.$$

Then Γ_1 is the discrete subgroup of $\mathcal{H}$ consisting of all integral matrices and Γ_p is a lattice of $\mathcal{H}$ containing Γ_1 with index p. Clearly

$$\mathrm{H}_1(\mathcal{H}/\Gamma_p; \mathbb{Z}) = \Gamma_p/[\Gamma_p, \Gamma_p] = \mathbb{Z}^2 \oplus \mathbb{Z}_p.$$

Note that these ${\Gamma_p}'$s produce infinitely many distinct nilmanifolds $\mathcal{N}_p = \mathcal{H}/\Gamma_p$ covered by $\mathcal{N}_1$. In this paper, we shall find all possible finite groups acting freely on each $\mathcal{N}_p$ by utilizing the method used in [1] and classify all such group actions, up to topological conjugacy. We shall use all notations and most of the Introduction, Section 2 and Section 3 of [1]. This work supplies missing one in [1, Theorem 3.11.].

Let $\pi = \langle t_1, t_2, t_3, | \; [t_2, t_1] = t_3^n, \;\; [t_3, t_1] = [t_3, t_2] = 1 \rangle$ be an almost Bieberbach group and N be a normal nilpotent subgroup of π with $G = \pi/N$ finite. For the almost Bieberbach group π, we find all normal nilpotent subgroups N of π, and classify (N, π) up to affine conjugacy.

2. Free actions of finite groups on the 3-dimensional nilmanifold for Type 1

Now we shall find all possible finite groups acting freely (up to topological conjugacy) on the 3-dimensional nilmanifold $\mathcal{N}_p$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\pi$. This was done by the program MATHEMATICA[10] and hand-checked.

Lemma 1. *Let N be a normal nilpotent subgroup of π and isomorphic to Γ_p. Then N can be represented by a sets of generators*

$$N = \langle t_1^{d_1} t_2^{m} t_3^{n_1}, \; t_2^{d_2} t_3^{n_2}, \; t_3^{\frac{n d_1 d_2}{p}} \rangle,$$

where d_1 and d_2 are divisors of p, and

$$0 \le m < \bar{d} = gcd(d_1, d_2), \quad 0 \le n_i < \frac{n d_1 d_2}{p}, \quad \frac{pm}{d_1 d_2} \in \mathbb{Z}.$$

Proof. Recall that $\pi = \langle t_1, t_2, t_3, \mid [t_2, t_1] = t_3^n,\ [t_3, t_1] = [t_3, t_2] = 1\rangle$. Let N be a normal nilpotent subgroup of π and isomorphic to Γ_p. Then by Proposition 3.1 in [1], we have

$$N = \langle t_1^{d_1} t_2^m t_3^\ell,\ t_2^{d_2} t_3^r,\ t_3^{\frac{nd_1d_2}{p}} \rangle, \quad \left(0 \le m < d_2,\ 0 \le \ell, r < \frac{nd_1d_2}{p}\right).$$

Recall that the normalizer $N_{\mathrm{Aff}(\mathcal{H})}(\pi)$ of π has been obtained [1, Theorem 3.11]:

$$N_{\mathrm{Aff}(\mathcal{H})}(\pi_1) = \left\{ \left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \right) \right\},$$

where $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{Z})$, if $ad - bc = 1$, then

$$x + u = \frac{1}{2}ab + \frac{k}{n}, \quad y + v = -\frac{1}{2}cd + \frac{k'}{n} \quad (k,\ k' \in \mathbb{Z}),$$

if $ad - bc = -1$, then

$$x + u = -\frac{1}{2}ab + \frac{k}{n}, \quad y + v = \frac{1}{2}cd + \frac{k'}{n} \quad (k,\ k' \in \mathbb{Z}).$$

Let $\bar{d} = \gcd(d_1,\ d_2)$. Then there exist $s,\ t \in \mathbb{Z}$ such that $\bar{d} = sd_1 + td_2$. Also there exist $q,\ w \in \mathbb{Z}$ such that $m = \bar{d}q + w \quad (0 \le w < \bar{d})$. Thus we have $\bar{d}q = sqd_1 + tqd_2$. Therefore it is not hard to see that

$$N \sim \langle {t_1}^{d_1} t_2^{m - sqd_1} t_3^{\ell'},\ t_2^{d_2} t_3^r,\ t_3^{\frac{nd_1d_2}{p}} \rangle = \langle {t_1}^{d_1} t_2^w t_3^{\ell''},\ t_2^{d_2} t_3^r,\ t_3^{\frac{nd_1d_2}{p}} \rangle,$$

by using

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{sq}{2} \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -sq & 1 \end{bmatrix} \right) \right) \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_1).$$

If we set $n_1 = \ell''$ and $n_2 = r$, then we have

$$N \quad \sim \quad \langle {t_1}^{d_1} t_2^w t_3^{n_1},\ t_2^{d_2} t_3^{n_2},\ t_3^{\frac{nd_1d_2}{p}} \rangle.$$

Note that the relation $t_1 (t_1^{d_1} t_2^m t_3^\ell) t_1^{-1} = (t_1^{d_1} t_2^m t_3^\ell)(t_3^{\frac{nd_1d_2}{p}})^{(-\frac{pm}{d_1d_2})} \in N$ shows that $\frac{pm}{d_1d_2} \in \mathbb{Z}$. Therefore we have proved the lemma. $\square$

Remark. The condition $\frac{pm}{d_1 d_2} \in \mathbb{Z}$ in the above lemma is crucial to determine the number of affinely non-conjugacy classes when $d_1,\ d_2$ and p are given. In fact, for $\bar{d} = (d_1, d_2)$ and $p = kD$, where D is the least common multiple of d_1 and d_2, we have $\frac{pm}{d_1 d_2} \in \mathbb{Z}$ if and only if $\frac{km}{\bar{d}} \in \mathbb{Z}$. Let $q = (\bar{d}, k)$. Then $\frac{km}{\bar{d}} \in \mathbb{Z}$ if and only if $\frac{k'm}{\bar{d}'} \in \mathbb{Z}$, where $k = qk',\ \bar{d} = q\bar{d}',\ (k',\bar{d}') = 1$. Thus $\bar{d}'$ is a divisor of m. Since $0 \le m < \bar{d} = q\bar{d}'$, we can get

$$m = 0, \bar{d}', \cdots, (q-1)\bar{d}'.$$

Theorem 2. *Let N^m and $N^{m'}$ be normal nilpotent subgroups of π and isomorphic to Γ_p whose sets of generators are*

$$N^m = \langle\, {t_1}^{d_1} t_2^m t_3^\ell,\ t_2^{d_2} t_3^r,\ t_3^{\frac{n d_1 d_2}{p}} \,\rangle, \quad N^{m'} = \langle\, {t_1}^{d_1} t_2^{m'} t_3^{\ell'},\ t_2^{d_2} t_3^{r'},\ t_3^{\frac{n d_1 d_2}{p}} \,\rangle.$$

If $m \neq m'$, then N^m is not affinely conjugate to $N^{m'}$.

Proof. Assume that N^m is affinely conjugate to $N^{m'}$. Then there exists

$$\mu = \left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \right) \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_1)$$

satisfying either

$$(*) \qquad \mu({t_1}^{d_1} {t_2}^m t_3^\ell)\mu^{-1} = t_1^{d_1} {t_2}^{m'} t_3^{\ell'}, \quad \mu(t_2^{d_2} t_3^r)\mu^{-1} = t_2^{d_2} t_3^{r'},$$

or

$$(**) \qquad \mu({t_1}^{d_1} {t_2}^m t_3^\ell)\mu^{-1} = t_2^{d_2} {t_3}^{r'}, \quad \mu(t_2^{d_2} t_3^r)\mu^{-1} = t_1^{d_1} {t_2}^{m'} t_3^{\ell'}.$$

From $(*)$, we obtain that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$, and $cd_1 = m - m' \neq 0$. Note that $0 \le m,\ m' < \bar{d}$ by Lemma 1. Since $d_1 \le |c| d_1 = |m - m'| < \bar{d} \le d_1$, we have a contradiction. However in $(**)$, we obtain the following relations:

$$bd_2 = d_1, \quad dd_2 = m', \quad ad_1 + bm = 0, \quad cd_1 + dm = d_2.$$

The relation $dd_2 = m' < d_2$ induces $d = 0,\ m' = 0$ and $bc = 1$. Therefore the relation $|a| d_1 = |-bm| = |m| < \bar{d} \le d_1$ implies $m = 0$, which is a contradiction. □

Let $N = \langle\, {t_1}^{d_1} t_2^m t_3^\ell,\ t_2^{d_2} t_3^r,\ t_3^{\frac{n d_1 d_2}{p}} \,\rangle$ be a normal nilpotent subgroup of π. The following theorem shows the conditions of affine conjugacy to N for given $d_1,\ d_2$ and m.

THEOREM 3. *Let N and N' be normal nilpotent subgroups of π whose sets of generators are*

$$N = \langle\, {t_1}^{d_1} t_2^m t_3^\ell,\ t_2^{d_2} t_3^r,\ t_3^{\frac{nd_1d_2}{p}}\ \rangle, \quad N' = \langle\, {t_1}^{d_1} t_2^m t_3^{\ell'},\ t_2^{d_2} t_3^{r'},\ t_3^{\frac{nd_1d_2}{p}}\ \rangle.$$

Then $N \sim N'$ is equivalent to either

$$r \equiv r' \ \ (mod \ \ d_2), \quad \ell \equiv (\ell' + \frac{m(r-r')}{d_2}) \ \ (mod \ \ d_1),$$

or $m = 0,\ d_1 = d_2$ *and d_1 is a divisor of $\ell + r'$ and $r + \ell'$.*

Proof. Assume that N is affinely conjugate to N'. Then there exists $\mu \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_1)$ satisfying either

$(*)$ $\qquad \mu({t_1}^{d_1} {t_2}^m t_3^\ell)\mu^{-1} = t_1^{d_1} {t_2}^m t_3^{\ell'}, \quad \mu(t_2^{d_2} t_3^r)\mu^{-1} = t_2^{d_2} t_3^{r'},$

or

$(**)$ $\qquad \mu({t_1}^{d_1} {t_2}^m t_3^\ell)\mu^{-1} = t_2^{d_2} {t_3}^{r'}, \quad \mu(t_2^{d_2} t_3^r)\mu^{-1} = t_1^{d_1} {t_2}^m t_3^{\ell'}.$

From $(*)$, we obtain the following relations:

$$bd_2 = 0, \quad dd_2 = d_2, \quad ad_1 + bm = d_1, \quad cd_1 + dm = m.$$

Thus we obtain $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Using this, we can get

$$x + u = \frac{r - r'}{nd_2}, \quad y + v = -\frac{\ell - \ell'}{nd_1} + \frac{m(r - r')}{nd_1 d_2}.$$

Since $x + u$ and $y + v$ are multiples of $\frac{1}{n}$, we have

$$\frac{r - r'}{d_2} \in \mathbb{Z} \ \text{ and } \ \frac{\ell - \ell'}{d_1} - \frac{m(r - r')}{d_1 d_2} \in \mathbb{Z}.$$

Therefore we can conclude that

$$r \equiv r' \ \ (mod \ \ d_2), \quad \ell \equiv (\ell' + \frac{m(r-r')}{d_2}) \ \ (mod \ \ d_1).$$

The converse is easy by using

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{\ell' - \ell}{nd_1} \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} \frac{r - r'}{nd_2} \\ \frac{m(r-r')}{nd_1 d_2} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \right) \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_1).$$

However in $(**)$, we obtain the following relations:

$$bd_2 = d_1, \quad dd_2 = m, \quad ad_1 + bm = 0, \quad cd_1 + dm = d_2.$$

The relation $0 \le dd_2 = m < d_2$ induces $d = 0$ and $m = 0$. Thus we have

$$a = 0, \quad b = c = 1, \quad d_1 = d_2.$$

Using this, we can get $x + u = -\frac{\ell + r'}{nd_2}, \quad y + v = \frac{r+\ell'}{nd_1}$. Since $x + u$ and $y + v$ are multiples of $\frac{1}{n}$, we have

$$\frac{\ell + r'}{d_2} \in \mathbb{Z} \text{ and } \frac{r + \ell'}{d_1} \in \mathbb{Z}.$$

Therefore $d_1(= d_2)$ is a divisor of $\ell + r'$ and $r + \ell'$. The converse is easy by

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} -\frac{\ell+r'}{nd_1} \\ \frac{r+\ell'}{nd_1} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \right) \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_1). \qquad \square$$

According to the above Remark, Theorems 2 and 3, we obtain the following result, which corrects the error in [1, Theorem 3.11.].

Corollary 4. *Let N be a normal nilpotent subgroup of π whose set of generators is*

$$N = \langle\, {t_1}^{d_1} t_2^m t_3^\ell,\ t_2^{d_2} t_3^r,\ t_3^{\frac{nd_1d_2}{p}} \,\rangle.$$

If $n \geqslant \frac{p}{min\{d_1, d_2\}}$, *then the number of affine conjugacy classes of normal nilpotent subgroups is*

$$\left\{ \begin{array}{ll} qd_1d_2 & \text{if } d_1 \neq d_2, \\ d_1d_2 - \dfrac{d_1}{2} + 1 & \text{if } d_1 = d_2,\ m = 0, \quad d_1 \in 2\mathbb{N}, \\ d_1d_2 - \dfrac{d_1 - 1}{2} & \text{if } d_1 = d_2,\ m = 0, \quad d_1 \in 2\mathbb{N} - 1, \end{array} \right\}$$

where $q = (gcd(d_1, d_2), k)$, *k can be obtained from $p = kD$ and D is the least common multiple of* d_1 *and* d_2. $\square$

Example. Assume $\mathbb{Z}_2 \times \mathbb{Z}_n$ acts freely on the nilmanifold $\mathcal{N}_2 = \mathcal{H}/\Gamma_2$ which yields an orbit manifold homeomorphic to $\mathcal{H}/\pi$. Then there exist 2 distinct affine conjugacy classes of free actions:

$$N_1 = \langle\, {t_1}^2,\ t_2,\ t_3^n \,\rangle, \quad N_2 = \langle\, {t_1}^2 t_3,\ t_2,\ t_3^n \,\rangle.$$

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