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CMC SURFACES WITH CONSTANT CONTACT ANGLE ALONG A CIRCLE

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Abstract. In this paper, we give a characterization of a Delaunay surface in $\mathbb{R}^3$. Let Σ be a CMC-H surface in $\mathbb{R}^3$ with $H \neq 0$. If Σ meets a plane with constant contact angle along a circle, then it is rotationally symmetric, i.e., Σ is part of a Delaunay surface.

1. Introduction

Björling first considered the problem to find a minimal surface containing a given real-analytic curve in its interior with the prescribed tangent planes. Known as the Björling problem, this was proved explicitly by Schwarz. Specifically, let $\gamma : J \to \mathbb{R}^3$ be a regular real-analytic curve defined on an interval J and $n : J \to \mathbb{R}^3$ be a real-analytic vector field along γ with $||n|| = 1$ and $\langle \gamma', n\rangle = 0$. Then there is a simply connected domain D containing J, on which the unique analytic extension $\tilde{\gamma}$ (resp. $\tilde{n}$) $: D \to \mathbb{R}^3$ of γ (resp. n) exists, such that a map $X : D \to \mathbb{R}^3$ defined by

$$X(u, v) = \mathrm{Re}\left(\gamma(z) - i\int_{z_0}^{z} \tilde{n}(w) \times \tilde{\gamma}'(w)\, dw\right), \tag{1.1}$$

where $z = u + iv \in D$, $z_0 \in J$, represents the unique minimal immersion such that $X|_J = \gamma$ and $n \perp X$ along γ. Using (1.1), Schwarz obtained symmetry principles for a minimal surface Σ as follows: (a) If Σ intersects a plane orthogonally, then there is a reflection symmetry with respect to the plane. (b) If Σ contains a straight line, then there is a rotation symmetry with respect to the straight line. The formula (1.1) has long been used to find examples of minimal surfaces. On the other

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hand, Pyo [6] obtained a characterization of a catenoid by using (1.1) as follows.

THEOREM ([6]). Let Σ be an immersed minimal surface in $\mathbb{R}^3$. If Σ meets a plane with constant contact angle along a circle, then it is part of a catenoid.

The holomorphicity of the Gauss map plays an important role in the Weierstrass representation formula for a minimal surface and hence in the formula (1.1). Although the Gauss map of a non-minimal CMC surface is just harmonic, Dorfmeister-Pedit-Wu [3] obtained a Weierstrass type representation formula for CMC surfaces in $\mathbb{R}^3$, which is called the DPW method. Any immersion in $\mathbb{R}^3$ with the constant mean curvature can be constructed from a Lie(ΛSL$(2,\mathbb{C})$)-valued holomorphic 1-form

$$\hat{\xi} = \sum_{j=-1}^{\infty} A_j \lambda^j dz,$$

where ΛSL$(2,\mathbb{C})$ is the loop group of maps $\phi : \mathbb{S}^1 \to \mathrm{SL}(2,\mathbb{C})$ with a twisting condition and Lie(ΛSL$(2,\mathbb{C})$) is the Lie algebra of the loop group ΛSL$(2,\mathbb{C})$. Motivated from Schwarz's result, for given real-analytic Björling data $\{\gamma,\nu\}$, and a non-zero constant H, Brander-Dorfmeister [2] proved the Björling problem for non-minimal CMC surfaces by using the DPW method.

THEOREM ([2]). Let $\gamma : J \to \mathbb{R}^3$ be a regular real-analytic curve and $\nu : J \to \mathbb{R}^3$ be a non-vanishing real-analytic vector field along γ such that $\langle \nu, \gamma' \rangle = 0$ along γ. Let H be a non-zero real number.

There is a CMC-H immersion $X : D \to \mathbb{R}^3$, where D is some open subset of $\mathbb{C}$ containing J, such that the restriction $X|_J$ coincides with γ, and such that the tangent planes to the immersion along γ are spanned by ν and γ'.

Moreover, the surface X is unique in the following sense: If $\widetilde{X}$ is any other solution, then, for every point $x_0 \in J$, there exists a neighborhood $N = (x_0 - \epsilon, x_0 + \epsilon) \times (-\delta, \delta) \subset \mathbb{C}$ of $z_0 = (x_0, 0) \in D$ such that $X|_N = \widetilde{X}|_N$.

In this paper, we deal with a characterization of a Delaunay surface in $\mathbb{R}^3$ analogous to the result obtained by Pyo [6]. Let Σ be a surface in $\mathbb{R}^3$ with the constant mean curvature $H \neq 0$. Suppose that Σ meets a plane with constant contact angle along a circle. Then we can compute the extended frame for Σ by using the method in [2], and hence we have the following result.

THEOREM 1.1. *Let Σ be a* CMC-H *surface in $\mathbb{R}^3$ with $H \neq 0$. If Σ meets a plane with constant contact angle along a circle, then it is rotationally symmetric, i.e., Σ is part of a Delaunay surface.*

2. Preliminaries

In this section, we give some basic notions and briefly introduce the construction of a CMC surface via integrable system method. We mainly refer to [1, 2, 4].

Let D be a simply connected domain in $\mathbb{R}^2$. Let Σ be a surface in $\mathbb{R}^3$ and $X : D \to \mathbb{R}^3$ be a conformal immersion of Σ with the metric $ds^2 = 4e^{2\varphi}(du^2 + dv^2)$. Let $z = u + iv$ be the canonical complex coordinate on $D \subset \mathbb{C} \simeq \mathbb{R}^2$. Then

$$\langle X_z, X_z\rangle = \langle X_{\bar z}, X_{\bar z}\rangle = 0 \text{ and } \langle X_z, X_{\bar z}\rangle = 2e^{2\varphi}. \tag{2.1}$$

The mean curvature of Σ is defined by

$$H = \frac{1}{2}(\kappa_1 + \kappa_2),$$

where κ_1 and κ_2 are principal curvatures of Σ. A surface Σ is said to be a *constant mean curvature surface* if H is constant, simply we call it a CMC *surface*, or a CMC-H *surface* when we emphasize the value H. Denote the unit normal vector field of Σ by $\mathrm{n} = \frac{X_u \times X_v}{|X_u \times X_v|}$. It is well known that $\triangle_\Sigma X = 2H\mathrm{n}$, and hence

$$H = \frac{1}{8}e^{-2\varphi}\langle X_{z\bar z}, \mathrm{n}\rangle.$$

Define the Hopf differential Q as

$$Q = \langle X_{zz}, \mathrm{n}\rangle.$$

From these, we can compute that

$$X_{zz} = 2\varphi_z X_z + Q\mathrm{n},\ X_{\bar z\bar z} = 2\varphi_{\bar z} X_{\bar z} + \bar Q\mathrm{n},\ X_{z\bar z} = 2He^{2\varphi}\mathrm{n}. \tag{2.2}$$

The Lie group SU(2) is a matrix group consists of all 2×2 unitary matrices

$$\begin{aligned}\mathrm{SU}(2) &= \{A \in \mathrm{GL}(2,\mathbb{C}) |\ AA^{\mathrm{H}} = I,\ \det A = 1\} \\ &= \left\{ \begin{pmatrix} a & b \\ -\bar b & \bar a \end{pmatrix} \,\middle|\, a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1 \right\},\end{aligned}$$

where $A^{\mathrm{H}} = \bar{A}^{\mathrm{T}}$ is the conjugate transpose of A. Denoted by $\mathfrak{su}(2)$ the Lie algebra of SU(2). It is a 3-dimensional real vector space consists of 2×2 traceless skew-Hermitian complex matrices:

$$\mathfrak{su}(2) = \left\{ \sigma \in \mathfrak{gl}(2, \mathbb{C}) \,\middle|\, \sigma + \sigma^{\mathrm{H}} = 0,\ \mathrm{tr}\sigma = 0 \right\}.$$

As a basis, take the following three matrices:

$$\sigma_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

There is an isometry between $\mathbb{R}^3$ and $\mathfrak{su}(2)$ that maps $(x, y, z) \in \mathbb{R}^3$ to the following matrix in $\mathfrak{su}(2)$

$$x\sigma_1 + y\sigma_2 + z\sigma_3 = \begin{pmatrix} iz & y - ix \\ -y - ix & -iz \end{pmatrix}, \tag{2.3}$$

with the metric $\langle \sigma, \tau \rangle = -\frac{1}{2}\mathrm{tr}(\sigma\tau)$ for any $\sigma, \tau \in \mathfrak{su}(2)$. In particular, $\langle \sigma_j, \sigma_k \rangle = \delta_{jk}$ for all j, k. From now on, we identify $\mathbb{R}^3$ with $\mathfrak{su}(2)$.

Note that $\{X_u, X_v, \mathrm{n}\}$ forms an orthogonal frame of $\Sigma \subset \mathbb{R}^3 \simeq \mathfrak{su}(2)$. Denote a SU(2)-valued frame by $F : D \to \mathrm{SU}(2)$ such that

$$F\sigma_1 F^{-1} = \frac{X_u}{|X_u|},\ F\sigma_2 F^{-1} = \frac{X_v}{|X_v|},\ F\sigma_3 F^{-1} = \mathrm{n}. \tag{2.4}$$

It yields that

$$X_z = -2ie^{\varphi} F \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} F^{-1},\ X_{\bar{z}} = -2ie^{\varphi} F \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} F^{-1}.$$

By choosing coordinates in $\mathbb{R}^3$, we may assume that $F(z_0) = I$ for a fixed point $z_0 \in D$. Differentiating X_z and $X_{\bar{z}}$, with the equations in (2.2), the $\mathfrak{su}(2)$-valued Maurer-Cartan form for F,

$$\omega = F^{-1}dF = U dz + V d\bar{z},$$

can be computed as follows (see [1, 2, 4]):

$$\begin{aligned} U &= F^{-1}F_z = \frac{1}{2}\begin{pmatrix} \varphi_z & -2e^{\varphi}H \\ e^{-\varphi}Q & -\varphi_z \end{pmatrix}, \\ V &= F^{-1}F_{\bar{z}} = \frac{1}{2}\begin{pmatrix} -\varphi_{\bar{z}} & -e^{-\varphi}\bar{Q} \\ 2e^{\varphi}H & \varphi_{\bar{z}} \end{pmatrix}. \end{aligned} \tag{2.5}$$

The compatibility condition $U_{\bar{z}} - V_z - [U, V] = 0$, which is equivalent to the Maurer-Cartan equation $d\omega + \omega \wedge \omega = 0$, can be written as

$$\varphi_{z\bar{z}} + e^{2\varphi} H^2 - \frac{1}{4} e^{-2\varphi} |Q|^2 = 0; \qquad \text{(Gauss equation)}$$

$$Q_{\bar{z}} = 2e^{2\varphi} H_z, \qquad \text{(Codazzi equation)} \tag{2.6}$$

where $[U, V] = UV - VU$.

Let $\lambda \in \mathbb{S}^1$ be a spectral parameter. Denoted by $\Lambda\mathrm{SU}(2)$ the loop group of maps $\phi : \mathbb{S}^1 \to \mathrm{SU}(2)$ with a twisting condition $\phi(-\lambda) = \sigma_3 \phi(\lambda) \sigma_3$: ϕ is an even (resp. odd) function in λ on its diagonal (resp. off-diagonal). Let $\mathrm{Lie}(\Lambda\mathrm{SU}(2))$ be the Lie algebra of $\Lambda\mathrm{SU}(2)$. Define a $\mathrm{Lie}(\Lambda\mathrm{SU}(2))$-valued 1-form $\hat{\omega}$, by adding a spectral parameter λ to ω, as follows.

$$\hat{\omega} = \hat{U} dz + \hat{V} d\bar{z},$$

where

$$\hat{U} = \frac{1}{2} \begin{pmatrix} \varphi_z & -2e^{\varphi} H \lambda^{-1} \\ e^{-\varphi} Q \lambda^{-1} & -\varphi_z \end{pmatrix}, \quad \hat{V} = \frac{1}{2} \begin{pmatrix} -\varphi_{\bar{z}} & -e^{-\varphi} \bar{Q} \lambda \\ 2e^{\varphi} H \lambda & \varphi_{\bar{z}} \end{pmatrix}. \tag{2.7}$$

Then $\hat{\omega}$ satisfies the Maurer-Cartan equation for all $\lambda \in \mathbb{S}^1$ if and only if Σ is a CMC surface in $\mathbb{R}^3$. More precisely, the following theorem holds.

Theorem ([2, 4]). Let $X : D \to \mathbb{R}^3$ be a conformal immersion. Then the mean curvature H is constant if and only if there is an extended frame $\hat{F}$ and the Maurer-Cartan 1-form $\hat{\omega} = \hat{F}^{-1} d\hat{F}$ such that $d\hat{\omega} + \hat{\omega} \wedge \hat{\omega} = 0$ for $\lambda \in \mathbb{S}^1$.

Here, $\hat{F} : D \to \Lambda\mathrm{SU}(2)$ is said to be an *extended frame* for a CMC surface if it is obtained by integrating $\hat{\omega}$ with the initial condition $\hat{F}(z_0) = I$ for $\lambda \in \mathbb{S}^1$, and $\hat{F}|_{\lambda=1} = F$.

Bobenko gave the expression for a CMC immersion in terms of an extended frame. For $H \neq 0$ and $\lambda \in \mathbb{S}^1$, the Sym-Bobenko formula is given by

$$\mathscr{S}_\lambda(\hat{F}) = -\frac{1}{2H} \left(\hat{F} \sigma_3 \hat{F}^{-1} + 2i\lambda (\partial_\lambda \hat{F}) \hat{F}^{-1} \right).$$

Theorem ([1, 2, 4]). Let $X : D \to \mathbb{R}^3$ be a CMC-H immersion. Let $\hat{F} : D \to \Lambda\mathrm{SU}(2)$ be an extended frame described as above. Then the immersion X can be written as

$$X(z) = \mathscr{S}_1(\hat{F}(z)) - \mathscr{S}_1(\hat{F}(z_0)) + X(z_0).$$

Conversely, for any φ and Q satisfying (2.6), if $\hat{F} \in \Lambda\mathrm{SU}(2)$ is a solution of the system $\hat{F}^{-1}\hat{F}_z = \hat{U}$ and $\hat{F}^{-1}\hat{F}_{\bar{z}} = \hat{V}$, where $\hat{U}$ and $\hat{V}$ are given as in (2.7), with $\det\hat{F} = 1$, then the Sym-Bobenko formula $\mathscr{S}_\lambda(\hat{F})$ describes a conformal CMC-H immersion into $\mathbb{R}^3$ with metric $ds^2 = 4e^{2\varphi}(du^2+dv^2)$ and the Hopf differential $\lambda^{-2}Q$.

Dorfmeister-Pedit-Wu [3] obtained a Weierstrass type representation formula for CMC surfaces in $\mathbb{R}^3$: Any CMC immersion in $\mathbb{R}^3$ can be constructed from a Lie($\Lambda\mathrm{SL}(2,\mathbb{C})$)-valued holomorphic 1-form

$$\hat{\xi} = \sum_{j=-1}^{\infty} A_j \lambda^j dz \text{ with } A_{-1} = \begin{pmatrix} 0 & a_{-1} \\ b_{-1} & 0 \end{pmatrix},\ a_{-1} \neq 0.$$

where $\Lambda\mathrm{SL}(2,\mathbb{C})$ is the loop group of maps $\phi : \mathbb{S}^1 \to \mathrm{SL}(2,\mathbb{C})$ with a twisting condition and Lie($\Lambda\mathrm{SL}(2,\mathbb{C})$) is the Lie algebra of the loop group $\Lambda\mathrm{SL}(2,\mathbb{C})$. We call $\hat{\xi}$ a *holomorphic potential*.

In this regard, Brander-Dorfmeister [2] proved the Björling problem for non-minimal CMC surfaces via DPW method. If a solution of the Björling problem exists, then the extension F of F_0 satisfies (2.1), (2.4) and (2.5). Therefore we use the conditions (2.1), (2.4) and (2.5) as necessary conditions for the existence of the extended frame $\hat{F}_0$ along J. We summarize the construction in [2] as the following five steps:

1. Translate given real-analytic Björling data $\{\gamma, \nu\}$ in terms of $\mathfrak{su}(2)$;
2. Let F_0 be a frame on an interval J. Determine the conformal metric φ on J by using (2.1) and (2.4);
3. Construct the extended frame $\hat{F}_0$, a solution of $\hat{F}_0^{-1}d\hat{F}_0 = \hat{\omega}_0$ with the initial condition along J, where (2.5) determines $\hat{\omega}_0$;
4. Find a holomorphic extension $\hat{\omega}$, which is called the *boundary potential*, of $\hat{\omega}_0$ on a simply connected domain D containing J;
5. Apply the DPW method.

3. Proof of Theorem 1.1

Definition 3.1. Let $P \subset \mathbb{R}^3$ be a plane normal to n_P. We say that a surface Σ *meets* P *with constant contact angle* β *along a curve* γ if $\gamma = \Sigma \cap P$ and $\langle \mathrm{n}, \mathrm{n}_P \rangle = \cos\beta$ is constant along γ.

Proof of Theorem 1.1. Let P be a plane normal to $\mathrm{n}_P = (0, \sin\beta, \cos\beta)$ passing through the origin in $\mathbb{R}^3$, that is,

$$P = \{(x,y,z) \in \mathbb{R}^3 |\ y\sin\beta + z\cos\beta = 0\}.$$

Denoted by γ a circle of radius r centered at the origin that lies in P. Parametrize γ as follows.

$$\gamma(u) = r\left(\sin\frac{2u}{r}, \cos\beta\cos\frac{2u}{r}, -\sin\beta\cos\frac{2u}{r}\right),\ u \in J,$$

where J is an open interval such that $0 \in J$. Without loss of generality, we may assume that Σ meets a plane P with constant contact angle β along γ, by a rigid motion of $\mathbb{R}^3$. The conormal vector field ν of Σ along γ satisfies that

$$\langle \nu, \gamma' \rangle = 0,\ \langle \nu, \mathrm{n}_P \rangle = \sin\beta.$$

Since $\{\gamma, \gamma', \mathrm{n}_P\}$ are mutually orthogonal along γ, we have

$$\begin{aligned}\nu &= \cos\beta\frac{\gamma}{|\gamma|} + \sin\beta\mathrm{n}_P \\ &= \left(\cos\beta\sin\frac{2u}{r}, \sin^2\beta + \cos^2\beta\cos\frac{2u}{r}, \cos\beta\sin\beta(1-\cos\frac{2u}{r})\right).\end{aligned}$$

Note that γ and ν are both real analytic. We claim that the solution of the Björling problem with respect to the analytic data $\{\gamma, \nu\}$ described above and $H \neq 0$ is a Delaunay surface. If the claim holds, then the conclusion follows by the maximum principle for CMC surfaces (or by the uniqueness theorem of [2]).

From (2.3), we identify γ' and ν with matrices in $\mathfrak{su}(2)$ as follows.

$$\gamma'(u) = 2\begin{pmatrix} i\sin\beta\sin\frac{2u}{r} & -\cos\beta\sin\frac{2u}{r} - i\cos\frac{2u}{r} \\ \cos\beta\sin\frac{2u}{r} - i\cos\frac{2u}{r} & -i\sin\beta\sin\frac{2u}{r}\end{pmatrix},$$

$$\nu(u) = \begin{pmatrix} i\cos\beta\sin\beta(1-\cos\frac{2u}{r}) & \begin{matrix}\sin^2\beta + \cos^2\beta\cos\frac{2u}{r} \\ -i\cos\beta\sin\frac{2u}{r}\end{matrix} \\ \begin{matrix}-\sin^2\beta - \cos^2\beta\cos\frac{2u}{r} \\ -i\cos\beta\sin\frac{2u}{r}\end{matrix} & -i\cos\beta\sin\beta(1-\cos\frac{2u}{r})\end{pmatrix}.$$

If there is a solution of the Björling problem, then there is a SU(2)-frame F satisfies (2.1) and (2.4). Thus, we let F_0 to be a frame along J such that

$$\begin{aligned}F_0\sigma_1F_0^{-1} &= \frac{1}{2e^{\varphi}}\gamma'(u), \\ F_0\sigma_2F_0^{-1} &= \nu(u),\end{aligned} \tag{3.1}$$

where the second equality follows from the necessary condition $X_v = 2e^{\varphi}\nu$ to make X to be a solution of the Björling problem. Taking the determinant to the first equality in (3.1) along J, we have

$$\varphi = \log\left(\frac{1}{2}\sqrt{\det(\gamma'(u))}\right) = 0. \tag{3.2}$$

Put $F_0 = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \in \mathrm{SU}(2)$. Then

$$\begin{aligned} F_0\sigma_1 F_0^{-1} &= \begin{pmatrix} -2i\mathrm{Re}(A\bar{B}) & -i(A^2 - B^2) \\ -i(\bar{A}^2 - \bar{B}^2) & 2i\mathrm{Re}(A\bar{B}) \end{pmatrix}, \\ F_0\sigma_2 F_0^{-1} &= \begin{pmatrix} 2i\mathrm{Im}(A\bar{B}) & A^2 + B^2 \\ -\bar{A}^2 - \bar{B}^2 & -2i\mathrm{Im}(A\bar{B}) \end{pmatrix}. \end{aligned}$$

The equations (3.1) and (3.2) yield that

$$\begin{aligned} 2\mathrm{Re}(A\bar{B}) &= -\sin\beta\sin\frac{2u}{r}, \\ 2\mathrm{Im}(A\bar{B}) &= \cos\beta\sin\beta\left(1 - \cos\frac{2u}{r}\right), \\ A^2 - B^2 &= \cos\frac{2u}{r} - i\cos\beta\sin\frac{2u}{r}, \\ A^2 + B^2 &= \sin^2\beta + \cos^2\beta\cos\frac{2u}{r} - i\cos\beta\sin\frac{2u}{r}. \end{aligned}$$

With the initial condition $F_0(0) = I$, the unique SU(2) frame F_0 is determined to be

$$F_0 = \begin{pmatrix} \cos\frac{u}{r} - i\cos\beta\sin\frac{u}{r} & -\sin\beta\sin\frac{u}{r} \\ \sin\beta\sin\frac{u}{r} & \cos\frac{u}{r} + i\cos\beta\sin\frac{u}{r} \end{pmatrix},$$

along J. Differentiating the frame F_0 with respect to u, we have

$$F_0^{-1}(F_0)_u = \frac{1}{r}\begin{pmatrix} -i\cos\beta & -\sin\beta \\ \sin\beta & i\cos\beta \end{pmatrix}.$$

For an extension F of F_0 satisfies (2.5),

$$F^{-1}F_u = U + V = \frac{1}{2}\begin{pmatrix} \varphi_z - \varphi_{\bar{z}} & -2e^{\varphi}H - e^{-\varphi}\bar{Q} \\ 2e^{\varphi}H + e^{-\varphi}Q & -\varphi_z + \varphi_{\bar{z}} \end{pmatrix}$$

and $F^{-1}F_u = F_0^{-1}(F_0)_u$ along J. Comparing these two values directly, along J,

$$\begin{aligned} \varphi_z &= -\frac{i}{r}\cos\beta, \\ Q &= \frac{2}{r}\sin\beta - 2H, \end{aligned}$$

because $\varphi_u = \varphi_z + \varphi_{\bar{z}} = 0$ along J. By (2.7), along J,

$$\hat{\omega}_0 = \frac{1}{2}\left\{\begin{pmatrix} -\frac{i}{r}\cos\beta & -2H\lambda^{-1} \\ (\frac{2}{r}\sin\beta - 2H)\lambda^{-1} & \frac{i}{r}\cos\beta \end{pmatrix} + \begin{pmatrix} -\frac{i}{r}\cos\beta & (-\frac{2}{r}\sin\beta + 2H)\lambda \\ 2H\lambda & \frac{i}{r}\cos\beta \end{pmatrix}\right\} du,$$

and hence the extended frame $\hat{F}_0$ can be determined by integrating $\hat{\omega}_0$ along J. Extend $\hat{\omega}_0$ holomorphically, we obtain the boundary potential as follows.

$$\hat{\omega} = \begin{pmatrix} -\frac{i}{r}\cos\beta & (H - \frac{1}{r}\sin\beta)\lambda - H\lambda^{-1} \\ H\lambda - (H - \frac{1}{r}\sin\beta)\lambda^{-1} & \frac{i}{r}\cos\beta \end{pmatrix} dz.$$

By Kilian [5], $\hat{\omega}$ coincides with the holomorphic potential of a Delaunay surface for any $r > 0$, β, and $H \neq 0$. This proves the claim. □

References

[1] A. I. Bobenko, *Surfaces in terms of* 2 *by* 2 *matrices. Old and new integrable cases*, Harmonic maps and integrable systems, Aspects Math., no. E23, Friedr. Vieweg, Braunschweig, (1994), 83-127.

[2] D. Brander, J. F. Dorfmeister, *The Björling problem for non-minimal constant mean curvature surfaces*, Comm. Anal. Geom., **18** (2010), no. 1, 171-194.

[3] J. F. Dorfmeister, F. Pedit, and H. Wu, *Weierstrass type representation of harmonic maps into symmetric spaces*, Comm. Anal. Geom., **6** (1998), no. 4, 633-668.

[4] S. Fujimori, S. Kobayashi, W. Rossman, *Loop group methods for constant mean curvature surfaces*, arXiv:math/0602570v2 [math.DG].

[5] M. Kilian, *On the associated family of Delaunay surfaces*, Proc. Amer. Math. Soc., **132** (2004), no. 10, 3075-3082.

[6] J. Pyo, *Minimal annuli with constant contact angle along the planar boundaries*, Geom. Dedicata., **146** (2010), 159-164.

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