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CONTROLLABILITY FOR SEMILINEAR STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DELAYS IN HILBERT SPACES

Daewook Kim* and Jin-Mun Jeong**

Abstract. In this paper, we investigate necessary and sufficient conditions for the approximate controllability for semilinear stochastic functional differential equations with delays in Hilbert spaces without the strict range condition on the controller even though the equations contain unbounded principal operators, delay terms and local Lipschitz continuity of the nonlinear term.

1. Introduction

In this paper, we study the approximate controllability for the following stochastic functional differential equations with delays in Hilbert spaces:

(1.1)
$$\begin{cases} dx(t) = [Ax(t) + \int_{-h}^{0} a(s)A_1x(t+s)ds + Bu(t)]dt + f(t, x_t)d\omega, \ t > 0, \\ x(0) = \phi^0 \in L^2(\Omega, H), \quad x(s) = \phi^1(s), s \in [-h, 0], \end{cases}$$

where $h > 0$, $a(\cdot)$ is Hölder continuous, $\omega(t)$ stands for K-valued Brownian motion or Winner process with a finite trace nuclear covariance operator Q, and g, f, are given functions satisfying some assumptions. Moreover, $A : D(A) \subset H \to H$ is unbounded and A_1 is a closed linear operator with domain containing that of A. Let U be a Banach space and the controller B be a linear bounded operator from U to H.

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** : the corresponding author.
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This kind of stochastic differential equations arises in many practical mathematical models, such as, option pricing, population dynamics, physical, biological and engineering problems, etc. (see [4, 6]). Many authors have studied for the theory of stochastic differential equations in a variety of ways in [1, 2] and reference therein. Recently, the existence results impulsive stochastic neutral differential equations have been studied by [7, 10], and impulsive neutral stochastic differential inclusions with nonlocal initial conditions by Lin and Hu [10, 12].

Most literature works have been devoted the approximate controllability for semilinear control systems with strict assumptions on the control action operator B. Moreover, in the previous works, the concept of fundamental solution (or Green function) is not used, so that the calculations to obtain the regularity and controllability conditions are complicated.

In this paper we investigate necessary and sufficient conditions for the approximate controllability for (1.1) without the strict range condition on B and the uniform boundedness in [16] even though the system (1.1) contains unbounded principal operators, delay terms and local Lipschitz continuity of the nonlinear term. For the basis of our study, we construct the fundamental solution for the linear systems(see [8, 13])and establish variations of constant formula of solutions for stochastic equation (1.1).

2. Preliminaries and lemmas

2.1. Retarded linear equations

The inner product and norm in H are denoted by $(\cdot,\cdot)$ and $|\cdot|$, respectively. V is another Hilbert space densely and continuously embedded in H. The notations $||\cdot||$ and $||\cdot||_*$ denote the norms of V and V^* as usual, respectively. For brevity we may regard that

$$||u||_* \leq |u| \leq ||u||, \quad u \in V. \tag{2.1}$$

Let A be the operator associated with the sesquilinear form satisfying

$$\mathrm{Re}\,(Au, u) \geq c_0||u||^2, \ u \in V. \tag{2.2}$$

Then A is a bounded linear operator from V to V^* according to the Lax-Milgram theorem, and A generates an analytic semigroup $S(t) = e^{tA}$ in both H and V^* as in Theorem 3.6.1 of [14]. Moreover, we have the following sequence

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*, \tag{2.3}$$

where each space is dense in the next one and continuous injection.

LEMMA 2.1. *With the notation (2.3), we have*

$$(V, V^*)_{1/2,2} = H, \quad (D(A), H)_{1/2,2} = V,$$

where $(V, V^*)_{1/2,2}$ *denotes the real interpolation space between* V *and* V^* *(Section 1.3.3 of* [15]*).*

First, we consider the following linear retarded functional differential equation with forcing term k:

$$\begin{cases} x'(t) = Ax(t) + \int_{-h}^{0} a(s)A_1x(t+s)ds + k(t), \quad t > 0, \\ x(0) = \phi^0, \quad x(s) = \phi^1(s) \quad -h \le s \le 0. \end{cases} \tag{2.4}$$

In order to construct the fundamental solution, we need to impose the following condition:

Assumption (A). The function $a(\cdot)$ is assumed to be real valued and Hölder continuous of order ρ in the interval $[-h, 0]$:

$$|a(s)| \le H_0, \quad |a(s) - a(\tau)| \le H_0(s-\tau)^\rho, \quad -h \le \tau, s \le 0 \tag{2.5}$$

for a constant H_0.

Let $W(\cdot)$ be the fundamental solution of the linear equation (2.4) in the sense of Nakaglri [11], which is the operator valued function satisfying

(2.6)
$$\begin{cases} W(t) &= S(t) + \int_0^t S(t-s)\{\int_{-h}^0 a(\tau)A_1W(s+\tau)d\tau\}ds, \quad t > 0, \\ W(0) &= I, \qquad W(s) = 0, \quad -h \le s < 0, \end{cases}$$

where $S(\cdot)$ is the semigroup generated by A. For each $t > 0$, we introduce the operator valued function $U_t(\cdot)$ defined by

$$U_t(s) = \int_{-h}^{s} W(t - s + \sigma)a(\sigma)A_1 d\sigma : V \to V, \quad s \in [-h, 0].$$

Then (2.4) is represented

$$x(t) = W(t)\phi^0 + \int_{-h}^{0} U_t(s)\phi^1(s)ds + \int_0^t W(t-s)k(s)ds.$$

From Proposition 4.1 of [8] or Theorem 1 of [13], it follows the following results.

LEMMA 2.2. *Under Assumption (A), the fundamental solution $W(t)$ to (2.4) exists uniquely and is bounded. Applying Proposition 4.1 of* [8] *to the equation (2.4), there exists a constant $C_0 > 0$ such that*

$$||W(t^{'}) - W(t)||_{\mathcal{L}(H)} \leq C_0(t^{'} - t), \tag{2.7}$$

$$||W(t^{'}) - W(t)||_{\mathcal{L}(V^*,V)} \leq C_0(t^{'} - t)^{\kappa}(t-h)^{-\kappa} \tag{2.8}$$

for $h < t < t^{'}$, and $\kappa < \rho$.

Let $T > 0$ be arbitrary fixed. Associated with $U_t(\cdot)$, we consider the operator $\mathcal{U} : L^2(-h, 0; V) \to L^2(0, T; V)$ defined by

$$(\mathcal{U}\phi^1)(t) = \int_{-h}^{0} U_t(s)\phi^1(s)ds, \quad t \in (0, T] \tag{2.9}$$

for $\phi^1 \in L^2(-h, 0; V)$. We can see that $\mathcal{U}$ is into and bounded for each $T > 0$(see [11]).

LEMMA 2.3. *Under Assumption (A) and $\kappa < \rho$, the operator $\mathcal{U}$ defined by (2.9) is Hölder continuous of order $(\kappa+1)/2$ in (h, ∞) in operator norm of $\mathcal{L}(H, V)$, i.e., for any $T > h$ there exists a constant C_T such that*

$$|(\mathcal{U}\phi^1)(t^{'}) - (\mathcal{U}\phi^1)(t)| \leq C_T|t^{'} - t|^{(\kappa+1)/2}. \tag{2.10}$$

Proof. Using $(V, V^*)_{1/2,2} = H$ and the well known interpolation inequality we get from (2.7) and (2.8)

$$||W(t^{'}) - W(t)||_{\mathcal{L}(V^*,H)} \leq C_0(t^{'} - t)^{(1+\kappa)/2}(t-h)^{-\kappa/2}. \tag{2.11}$$

Hence, with aid of (2.11) we have

$$\begin{aligned}
&|(\mathcal{U}\phi^1)(t^{'}) - (\mathcal{U}\phi^1)(t)| \\
&\leq C_0 \int_{-h}^{0}\int_{-h}^{s} (t^{'} - t)^{(\kappa+1)/2}(t - s + \sigma - h)^{-\kappa/2} H_0 ||A||_{\mathcal{L}(V^*,V)}||\phi^1(s)||d\sigma ds \\
&\leq C_0H_0\big(1 - \frac{\kappa}{2}\big)^{-1}\sqrt{h}||A||_{\mathcal{L}(V^*,V)}(t^{'} - t)^{(\kappa+1)/2}(t-h)^{1-\kappa/2}||\phi^1||_{L^2(-h,0;V)},
\end{aligned}$$

Noting that $t - s + \sigma > h$ since $t > h$, we get (2.10). □

By virtue of Lemma 2.1, we can follow the argument of Di Blasio et al. [3] term by term to deduce the following result.

PROPOSITION 2.4. *Assume that $(\phi^0, \phi^1) \in V \times L^2(-h, 0; V)$ and $k \in L^2(0, T; V^*)$ for $T > 0$. Then, there exists a solution x of the system (2.4) such that*

$$x \in L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H).$$

2.2. Semilinear stochastic differential equations

In this paper $(H, |\cdot|)$ and $(K, |\cdot|_K)$ denote real separable Hilbert spaces. Consider the following retarded semilinear stochastic control system in Hilbert space H:

(2.12)

$$\begin{cases} x^{'}(t) = Ax(t) + \int_{-h}^{0} a_1(s)A_1x(t+s)ds + f(t, x_t)d\omega + Bu(t), & t > 0, \\ x(0) = \phi^0 \in L^2(\Omega, H), \quad x(s) = \phi^1(s), s \in [-h, 0]. \end{cases}$$

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space furnished with complete family of right continuous increasing sub σ-algebras $\{\mathcal{F}_t, t \in I\}$ satisfying $\mathcal{F}_t \subset \mathcal{F}$.

An H valued random variables is an $\mathcal{F}$ -measurable function $x(t) : \Omega \to H$ and the collection of random variables $\mathcal{S} = \{x(t, w) : \Omega \to H : t \in [0, T], \ w \in \Omega\}$ is a stochastic process. Generally, we just write $x(t)$ instead of $x(t, w)$ and $x(t) : [0, T] \to H$ in the space of $\mathcal{S}$

Let $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormal basis of K, and let $Q \in B(K, K)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite $\mathrm{Tr}(Q) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} = \lambda < \infty$ (Tr denotes the trace of the operator), where $\lambda_n \geq 0 (n = 1, 2, \cdots)$, and $B(K, K)$ denotes the space of all bounded linea operators from K into K.

$\{\omega(t) : t \geq 0\}$ be a cylindrical K-valued Wiener process with a finite trace nuclear covariance operator Q over $(\Omega, \mathcal{F}, P)$, which satisfies that

$$\omega(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} w_i(t) e_n, \quad t \geq 0,$$

where $\{w_i(t)\}_{i=1}^{\infty}$ be mutually independent one dimensional standard Wiener processes over $(\Omega, \mathcal{F}, P)$. Then the above K-valued stochastic process $\omega(t)$ is called a Q-Wiener process.

We assume that $\mathcal{F}_t = \sigma\{\omega(s) : 0 \leq s \leq t\}$ is the σ-algebra generated by w and $\mathcal{F}_T = \mathcal{F}$. Let $\psi \in B(K, H)$ and define

$$|\psi|_Q^2 = \mathrm{Tr}(\psi Q \psi^*) = \sum_{n=1}^{\infty} |\sqrt{\lambda_n} \psi e_n|^2.$$

If $|\psi|_Q^2 < \infty$, then ψ is called a Q-Hilbert-Schmidt operator. $B_Q(K, H)$ stands for the space of all Q-Hilbert-Schmidt operators. The completion $B_Q(K, H)$ of $B(K, H)$ with respect to the topology induced by the norm $|\psi|_Q$, where $|\psi|_Q^2 = (\psi, \psi)$ is a Hilbert space with the above norm topology.

Let V be a dense subspace of H as mentioned in Section 2.1. For $T > 0$ we define

$$M^2(-h,T;V) = \{x : [-h,T] \to V : E(\int_{-h}^{T} ||x(s)||^2 ds) < \infty\}.$$

The spaces $M^2(-h,0;V)$, $M^2(0,T;V)$, and $M^2(0,T;V^*)$ are also defined as the same way and the basic theory of M_2 can be founded in [5].

For $h > 0$, we assume that $\phi^1 : [-h,0) \to V$ is a given initial value satisfying

$$E(\int_{-h}^{0} ||\phi^1(s)||^2 ds) < \infty,$$

that is, $\phi^1 \in M^2(-h,0;V)$. In this note, a random variable $x(t) : \Omega \to H$ will be called an L^2-primitive process if $x \in M^2(-h,T;V)$.

Definition 2.5. A stochastic process $x : [-h,T] \times \Omega \to H$ is called a solution of (2.12) if

(i) $x(t)$ is measurable and $\mathcal{F}_t$-adapted for each $t \geq 0$.
(ii) $x(t) \in H$ has cádlág paths on $t \in (0,T)$ such that
(2.13)

$$x(t) = W(t)\phi^0 + \int_{-h}^{0} U_t(s)\phi^1(s)ds + \int_0^t W(t-s)\{f(s,x_s)d\omega + Bu(s)\}ds,$$

(iii) $x \in M^2(0,T;V)$ i.e., $E(\int_0^T ||x(s)||^2 ds) < \infty$ and $C([0,T];H)$.

To establish our results, we introduce the following assumptions on System (2.12). For each $s \in [0,T]$, we define $x_s : [-h,0] \to H$ as $x_s(r) = x(s+r), \quad -h \leq r \leq 0$. We will set

$$\Pi = M^2(-h,0;V).$$

Assumption (F). Let $f : \mathbb{R} \times \Pi \to B(K,H)$ be a nonlinear mapping satisfying the following: for every $t \in [0,T]$, $||x||_\Pi \leq r$, and $||y||_\Pi \leq r$

(i) For any $x \in \Pi$, the mapping $f(\cdot,x)$ is strongly measurable.
(ii) There exists a function $L_f : \mathbb{R}_+ \to \mathbb{R}$ such that

$$E|f(t,x) - f(t,y)|^2 \leq L_f(r)||x-y||_\Pi^2, \quad t \in [0,T].$$

(iii) The inequality

$$E|f(t,x)|^2 \leq L_f(r)(||x||_\Pi + 1)^2.$$

LEMMA 2.6. *Let $x \in M^2(-h,T;V)$. Then the mapping $s \mapsto x_s$ belongs to $C([0,T];\Pi)$, and for each $0 < t \le T$*

(2.14)
$$||x_t||_{\Pi} \le ||x||_{M^2(-h,t;V)} = ||\phi^1||_{\Pi} + ||x||_{M^2(0,t;V)},$$
$$E(||x||^2_{L^2(0,t;V)}) = ||x||^2_{M^2(0,t;V)},\ ||x.||_{L^2(0,t;\Pi)} \le \sqrt{t}||x||_{M^2(-h,t;V)}.$$

Proof. The first paragraph is easy to verify. In fact, it is from the following inequality;

(2.15)
$$||x_t||^2_{\Pi} = E\big(\int_{-h}^{0} ||x(t+\tau)||^2 d\tau\big) \le E\big[\int_{-h}^{t} ||x(\tau)||^2 d\tau\big] \le ||x||^2_{M^2(-h,t;V)}.$$

The second paragraph is immediately obtained by definition. From the above inequality, we have

$$\int_0^t ||x_s||^2_{\Pi} ds = \int_0^t \big[E\big(\int_{s-h}^{s} ||x(\tau||^2 d\tau\big)\big]^2 ds \le t||x||^2_{M^2(-h,t;V)},$$

which completes the last paragraph. □

The following results on the solvability of equation (2.12) is from [9].

PROPOSITION 2.7. *1) Let Assumptions (A) and (F) be satisfied. Assume that $(\phi^0,\phi^1) \in L^2(\Omega,H) \times \Pi$ and $k \in M^2(0,T;V^*)$ for $T > 0$. Then, there exists a solution x of the system (2.12) such that*

$$x \in M^2(0,T;V) \cap C([0,T];H) := \mathcal{Z}(T).$$

Moreover, there is a constant C_1 such that

$$||x||_{\mathcal{Z}(T)} \le C_1(1 + E(|\phi^0|^2) + ||\phi^1||_{\Pi} + ||k||_{M^2(0,T;V^*)}). \tag{2.16}$$

2) Let Assumptions (A) and (F) be satisfied. Assume that $(\phi^0,\phi^1) \in L^2(\Omega,V) \times M^2(-h,0;D(A))$ and $k \in M^2(0,T;H)$ for $T > 0$. Then, there exists a solution x of the system (2.12) such that

$$x \in M^2(0,T;D(A)) \cap C([0,T];V) := \mathcal{Z}_0(T).$$

Moreover, there is a constant C_1 such that

(2.17)
$$||x||_{\mathcal{Z}_0(T)} \le C_1(1 + E(||\phi^0||^2) + ||\phi^1||_{M^2(-h,0;D(A))} + ||k||_{M^2(0,T;H)}).$$

PROPOSITION 2.8. *Suppose that $k \in M^2(0,T;H)$ and $y(t) = \int_0^t W(t-s)k(s)ds$ for $0 \le t \le T$. Then there exists a constant C_2 such that*

$$||y||_{M^2(0,T;H)} \le C_2 T||k||_{M^2(0,T;H)}, \tag{2.18}$$

and

$$||y||_{M^2(0,T;V)} \le C_2\sqrt{T}||k||_{M^2(0,T;H)}. \tag{2.19}$$

Proof. It is easily obtained that

$$||y||_{M^2(0,T;H)} \le T\sqrt{C_0/2}||k||_{M^2(0,T;H)}. \tag{2.20}$$

From Lemma 2.1, (2.17), and (2.20) it holds that

$$||y||_{M^2(0,T;V)} \le M_0\sqrt{C_1 T}(C_0/2)^{1/4}||k||_{M^2(0,T;H)}.$$

So, if we take a constant $C_2 > 0$ such that

$$C_2 = \max\{\sqrt{C_0/2}, M_0\sqrt{C_1}(C_0/2)^{1/4}\},$$

the proof is complete. □

3. Approximately reachable sets

Let U be a Banach space and the controller operator B be bounded linear operator from another Banach space U to H. The solution $x(t) = x(t;\phi,F,u)$ of initial value problem (1,1) is the following form:

$$\begin{aligned} x(t;\phi,f,u) &= W(t)\phi^0 + \int_{-h}^{0} U_t(s)\phi^1(s)ds + \int_0^t W(t-s)\{f(s,x_s)d\omega \\ &\quad + Bu(s)ds\}, \\ U_t(s) &= \int_{-h}^{s} W(t-s+\sigma)a(\sigma)A_1 d\sigma. \end{aligned}$$

For $T > 0$, $\phi = (\phi^0,\phi^1) \in L^2(\Omega,H)\times \Pi$ and $u \in L^2(0,T;U)$ we define reachable sets as follows.

$$L_T(\phi) = \{z \in H : |x(T;\phi,0,u) - z| \le \epsilon, \quad \forall u \in L^2(0,T;U),\ \forall \epsilon > 0\},$$
$$R_T(\phi) = \{z \in H : E|x(T;\phi,f,u) - z| \le \epsilon, \quad \forall u \in L^2(0,T;U),\ \forall \epsilon > 0\}.$$

Here, we know that reachable sets is independent of the initial data.

THEOREM 3.1. *For any $T > 0$ we have*

$$R_T(0) \subset L_T(0).$$

Proof. Let $z_0 \notin L_T(0)$, where $L_T(0)$ is the reachable set of the linear system for the initial value $\phi = (0,0)$. Since $L_T(0)$ is a balanced closed convex subspace, we have $\alpha z_0 \notin L_T(0)$ for every $\alpha \in \mathbb{R}$, and

$$\inf\{|z_0 - z| : z \in L_T(0)\} = d.$$

By the formula (2.16) we have

$$(3.1) \qquad ||x(\cdot;0,f,u)||_{M^2(0,T;V)} \le C_1||Bu||_{M^2(0,T;H)},$$

where C_1 is the constant in Proposition 2.7. For every $u \in M^2(0,T;U)$, we choose a constant $\alpha > 0$ such that

$$(3.2) \qquad C_2\mathrm{Tr}(Q)\sqrt{T_1L_f(r)}(||\phi^1||_\Pi + C_1C_B||u||_{L^2(0,T;H)} + 1) < \alpha d.$$

By using Hölder inequality, we have

$$\begin{aligned}(3.3) \quad E\Big|\int_{T-\delta}^T W(T-s)f(s,x_s)d\omega\Big| &\le C_0\sqrt{\delta}\mathrm{Tr}(Q)L_f(r)(||x_s||_\Pi + 1)\\ &\le C_0\sqrt{\delta}\mathrm{Tr}(Q)L_f(r)(||\phi^1||_\Pi + ||x(\cdot;0,f,u)||_{M^2(0,T_1;V)} + 1).\end{aligned}$$

Hence form (3.1) and (3.2), it follows that

$$\begin{aligned} &E|x(T;0,f,u) - \alpha z_0|\\ &\quad \ge E\Big|\int_0^T W(T-s)Bu(s)ds - \alpha z_0\Big| - E\Big|\int_0^t S(t-s)f(s,x_s)d\omega\Big|\\ &\quad \ge \alpha d - C_2\mathrm{Tr}(Q)\sqrt{T_1L_f(r)}(||\phi^1||_\Pi + C_1||Bu||_{M^2(0,T;H)} + 1) > 0.\end{aligned}$$

Thus, we have $\alpha z_0 \notin R_T(0)$. □

We assume the following conditions:

Assumption (B1) For $0 \le \tau < t \le T$ and $u \in L^2(0,T;U)$, the $\mathcal{B}(\tau,t)$ from $L^2(0,T;U)$ into H defined by

$$\mathcal{B}(\tau,t)k := \int_\tau^t W(t-s)Bu(s)ds$$

induces an invertible operator $\hat{\mathcal{B}}(\tau,t)$ defined on $L^2(0,T;U)/\mathrm{Ker}\mathcal{B}(\tau,t)$ and there exists a positive constant L_B such that $||\hat{\mathcal{B}}(\tau,t)^{-1}|| \le L_B$, see [12].

Assumption (B2) For every $u \in L^2(0,T;U)$, there exists a constant C_B such that

$$||Bu||_{M^2(0,T;H)} \le C_B||u||_{L^2(0,T;U)}.$$

THEOREM 3.2. *Under Assumptions (A), (B1-2), (F), and $T > h$, we have*

$$\overline{L_T(\phi)} \subset \overline{R_T(\phi)}, \quad \phi \in L^2(\Omega,H) \times \Pi.$$

Proof. Let $T > h$, and let $\gamma > 0$ be arbitrary given. We will show that $z \in L_T(\phi)$ satisfying $|z| < \gamma$ belongs to $R_T(\phi)$. Let $u \in L^2(0,T;U)$ be arbitrary fixed. Then by Proposition 2.7, we have

$$||x_u||_{\mathcal{Z}(T)} \le C_1(1 + E(|\phi^0|^2) + ||\phi^1||_{\Pi} + C_B||u||_{L^2(0,T;H)}).$$

where x_u is the solution of (1.1) corresponding to the control u.

For any $\epsilon > 0$, we can choose a constant $\delta > 0$ satisfying

(3.4)
$$\begin{aligned}
&\max\{\delta, \sqrt{\delta}\}\\
&< \min\big\{\big(2\delta C_0 C_1\big(1 + E(|\phi^0|^2)\big)\big)^{-1},\\
&\quad \big(C_0||\mathcal{U}||_{\mathcal{L}(L^2(-h,0;V),L^2(0,T;V))}||\phi^1||_{\Pi}\big)^{-1},\ \big(C_0 C_T||\phi^1||_{\Pi}\big)^{-1},\\
&\quad \big((C_0+1)C_0\sqrt{T}C_2\mathrm{Tr}(Q)\sqrt{T_1 L_f(r)}\big(||\phi^1||_{\Pi} + ||x_u||_{\mathcal{Z}(T)} + 1\big)\big)^{-1},\\
&\quad \big(M_0 + \frac{\epsilon M_1}{8}\big)^{-1},\ \big((C_0+1)C_0 C_B T||u||_{L^2(0,T;U)}\big)^{-1}\big\}\epsilon/8,
\end{aligned}$$

where

$$\begin{aligned}
(3.5)\quad M_0 =& C_0 C_1 \mathrm{Tr}(Q) L_f(r)\big\{1 + E(|\phi^0|^2) + ||\phi^1||_{\Pi} + C_B||u||_{L^2(0,T;H)}\\
&+ C_B L_B\big(C_0||x_u||_{\mathcal{Z}(T)} + \gamma\big)\big\},\\
(3.6)\quad M_1 =& C_0 C_1 C_B L_B \mathrm{Tr}(Q) L_f(r).
\end{aligned}$$

Set

$$\begin{aligned}
x_1 := x(T-\delta;\phi,f,u) &= W(T-\delta)\phi^0 + \int_{-h}^0 U_{T-\delta}(s)\phi^1(s)ds\\
&+ \int_0^{T-\delta} W(T-\delta-s)f(s,(x_u)_s)d\omega + \int_0^{T-\delta} W(T-\delta-s)Bu(s)ds,
\end{aligned}$$

where $x_u(s) = x(s;\phi,f,u)$ and $(x_u)_s(\tau) = x(s+\tau;\phi,f,u)$ for $0 < t \le T$ and $-h \le \tau < 0$. Consider the following problem:

$$(3.7)\quad \begin{cases} y'(t) = Ay(t) + \int_{-h}^0 a(s)A_1 y(t+s)ds + Bu(t), & \delta < t \le T,\\ y(T-\delta) = x_1, \quad y(s) = 0 & -h \le s \le 0. \end{cases}$$

The solution of (3.7) with respect to the control $w \in L^2(T-\delta,T;U)$ is denoted by

$$(3.8)\qquad y_w(T) = W(\delta)x_1 + \int_{T-\delta}^T W(T-s)Bw(s)ds.$$

Then since $z \in L_T(\phi)$, and $L_T(\phi) = L(0)$ is independent of the time T and initial data ϕ, there exists $w_1 \in L^2(T-\delta, T; U)$ such that

$$|y_{w_1}(T) - z| < \frac{\epsilon}{8}. \tag{3.9}$$

and so, by (3.8) and (3.9) we have

$$\begin{aligned} |\int_{T-\delta}^{T} W(T-s)Bw_1(s)ds| &\le |y_{w_1}(T) - z| + \gamma + |W(\delta)x_1| \\ &\le C_0||x_u||_{\mathcal{Z}(T)} + \gamma + \frac{\epsilon}{8}. \end{aligned}$$

Hence, from Assumptions (B1) and (B2), it follows that

$$||Bw_1||_{M^2(0,T;H)} \le C_B L_B\big(C_0||x_u||_{\mathcal{Z}(T)} + \gamma + \frac{\epsilon}{8}\big) \tag{3.10}$$

Now we set

$$v(s) = \begin{cases} u & \text{if } 0 \le s \le T-\delta, \\ w_1(s) & \text{if } T-\delta < s < T. \end{cases}$$

Then $v \in M^2(0,T;U)$. Observing that

$$\begin{aligned} x_v(t;\phi,f,v) =& W(t)\phi^0 + \int_{-h}^{0} U_t(s)\phi^1(s)ds + \int_0^t W(t-\tau)\{f(\tau,(x_v)_\tau)dW \\ &+ Bv(\tau)d\tau\}, \end{aligned}$$

from (3.9) and (3.10) we obtain that

(3.11)

$$\begin{aligned} &E|x(T;\phi,G,v) - z| \le E|y_{w_1}(T) - z| + E\big|W(T)\phi^0 - W(\delta)W(T-\delta)\phi^0\big| \\ &+ E\big|\int_{-h}^{0} U_T(s)\phi^1(s)ds - W(\delta)\int_{-h}^{0} U_{T-\delta}(s)\phi^1(s)ds\big| \\ &+ E\big|\int_0^T W(T-s)f(s,(x_u)_s)dW - W(\delta)\int_0^{T-\delta} W(T-\delta-s)f(s,(x_u)_s)dW\big| \\ &+ E\big|\int_0^{T-\delta} W(T-s)Bu(s)ds - W(\delta)\int_0^{T-\delta} W(T-\delta-s)Bu(s)ds\big| \\ &\le \frac{\epsilon}{8} + I + II + III + IV. \end{aligned}$$

From (2.16) it follows that

$$\sup_{0\le t\le T} E|W(t)\phi^0| \le ||W(\cdot)\phi^0||_{\mathcal{Z}(T)} \le C_1\big(1 + E(|\phi^0|^2)\big),$$

and so

(3.12)

$$
\begin{aligned}
I &= E\big|W(T)\phi^0 - W(\delta)W(T-\delta)\phi^0\big| \\
&\le ||I - W(\delta)||_{\mathcal{L}(H)} E|W(T)\phi^0| + ||W(T) - W(T-\delta)||_{\mathcal{L}(H)}|W(\delta)\phi^0| \\
&\le 2\delta C_0 C_1\big(1 + E(|\phi^0|^2)\big) < \frac{\epsilon}{8}.
\end{aligned}
$$

With aid of Lemmas 2.2 and 2.3, and (3.4) we have

(3.13)

$$
\begin{aligned}
II =& E\Big|\int_{-h}^{0} U_T(s)\phi^1(s)ds - W(\delta)\int_{-h}^{0} U_{T-\delta}(s)\phi^1(s)ds\Big| \\
\le& E\Big|(I - W(\delta))\int_{-h}^{0} U_T(s)\phi^1(s)ds\Big| + E\Big|W(\delta)\int_{-h}^{0}(U_T(s) - U_{T-\delta}(s))\phi^1(s)ds\Big| \\
\le& \delta C_0 ||\mathcal{U}||_{\mathcal{L}(L^2(-h,0;V))}||\phi^1||_{\Pi} + C_0 C_T \delta^{(\kappa+1)/2}||\phi^1||_{\Pi} < \frac{\epsilon}{4}.
\end{aligned}
$$

By (2.16) and (3.10), we get

(3.14)

$$
\begin{aligned}
&||x_{w_1}||_{M^2(T-\delta,T;V)} \le ||x_v||_{M^2(0,T;V)} \\
\le& C_1\big\{1 + E(|\phi^0|^2) + ||\phi^1||_{\Pi} + C_B||u||_{L^2(0,T;H)} + C_B L_B\big(C_0||x_u||_{\mathcal{Z}(T)} + \gamma + \frac{\epsilon}{8}\big)\big\}.
\end{aligned}
$$

Hence, with aid of Assumption (F) and (3.14) and by using Hólder inequality, we have

(3.15)

$$
\begin{aligned}
&E\Big|\int_{T-\delta}^{T} W(T-s)f(s,(x_{w_1})_s)d\omega\Big| \le C_0\sqrt{\delta}\mathrm{Tr}(Q)||f(s,(x_{w_1})_s)||_{M^2(0,T;H)} \\
&\le C_0\sqrt{\delta}\mathrm{Tr}(Q)L_f(r)(||(x_{w_1})_s||_{\Pi} + 1) \\
&\le C_0\sqrt{\delta}\mathrm{Tr}(Q)L_f(r)(||\phi^1||_{\Pi} + ||x_{w_1}||_{M^2(0,T_1;V)} + 1) \le \sqrt{\delta}\big(M_0 + \frac{\epsilon M_1}{8}\big),
\end{aligned}
$$

where M_0 and M_1 are the constants of (3.5) and (3.6), respectively. Thus, from (2.7), (2.8), (3.3), (3.4), and (3.15) it follows that

(3.16)

$$
\begin{aligned}
III = & E\big|W(\delta)\int_0^{T-\delta} W(T-\delta-s)f(s,(x_u)_s)d\omega - \int_0^T W(T-s)f(s,(x_v)_s)d\omega\big| \\
\leq & E\big|(W(\delta)-I)\int_0^{T-\delta} W(T-\delta-s)f(s,(x_u)_s)d\omega\big| \\
& + E\big|\int_0^{T-\delta}\big(W(T-\delta-s)-W(T-s)\big)f(s,(x_u)_s)d\omega\big| \\
& + E\big|\int_{T-\delta}^{T} W(T-s)f(s,(x_{w_1})_s)d\omega\big| \\
\leq & (C_0+1)C_0\delta\sqrt{T}\mathrm{Tr}(Q)L_f(r)(||\phi^1||_{\Pi}+||x_u||_{\mathcal{Z}(T)}+1)+\sqrt{\delta}\big(M_0+\frac{\epsilon M_1}{8}\big) \\
< & \frac{\epsilon}{8}+\frac{\epsilon}{8}\leq\frac{\epsilon}{4},
\end{aligned}
$$

and

(3.17)

$$
\begin{aligned}
IV = & E\big|W(\delta)\int_0^{T-\delta} W(T-\delta-s)Bu(s)ds - \int_0^{T-\delta} W(T-s)Bu(s)ds\big| \\
\leq & E\big|(W(\delta)-I)\int_0^{T-\delta} W(T-\delta-s)Bu(s)ds\big| \\
& + E\big|\int_0^{T-\delta}\big(W(T-\delta-s)-W(T-s)\big)Bu(s)ds\big| \\
\leq & (C_0+1)C_0C_B\delta\sqrt{T}||u||_{L^2(0,T;U)} < \frac{\epsilon}{8}.
\end{aligned}
$$

Therefore, by (3.12)-(3.16), we have

$$||x(T;\phi,G,v)-z|| < \epsilon,$$

that is, $z \in \overline{R_T(\phi)}$ and the proof is complete. □

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Department of Mathematics Education
Seowon University
Chungbuk 28674, Republic of Korea
E-mail: kdw@seowon.ac.kr

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Department of Applied Mathematics
Pukyong National University
Busan 48513, Republic of Korea
E-mail: jmjeong@pknu.ac.kr