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VARIOUS REMARKS ON HOMOLOGICAL INVARIANTS OF LOCAL RINGS

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Abstract. In this article, we investigate the finiteness of Auslander Index when a ring A has not necessarily a canonical module, or a Gorenstein module. We also study the relations between column invariants and row invariants.

1. Backgrounds and preliminaries

Throughout this paper, we assume that $(A, \mathfrak{m})$ is a Noetherian local ring, and all modules are unitary.

In [3], Ding showed that when a Cohen-Macaulay local ring A has a canonical module, $Index(A)$ is finite if and only if $A_{\mathfrak{p}}$ is Gorenstein for all nonmaximal prime ideal $\mathfrak{p}$ (we say that A is Gorenstein on the punctured spectrum): $Index(A)$ is defined to be the smallest positive integer n such that $A/\mathfrak{m}^n$ is not a homomorphic image of a maximal Cohen-Macaulay module with no nonzero free summand. In [11], the author proved Ding's result under the weaker condition that A has a Gorenstein module, which may have type greater than one while a canonical module has always type one.

In this article, we investigate the condition for A to have a finite index when A has not necessarily a canonical module, or a Gorenstein module. We actually show that a Cohen-Macaulay local ring A of dimension d, not necessarily having a Gorenstein module, has a finite index provided that every maximal Cohen-Macaulay module is a $(d+1)$-st syzygy. To prove this fact, we use the properties of column invariant introduced in [5,6]. It is proved in [5,6] that there are certain restrictions on the entries of the maps in the minimal free resolutions of finitely generated modules of infinite projective dimension over Noetherian local rings. This fact provides not only a new way to understand some previously

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known results in commutative ring theory (see for instance [5,6]), but also new interesting invariants of local rings.

We also simplify the proofs of the previously known results about column-row invariants, provided that $col_M(A)$ is finite. In particular, we prove that $col(A)$ is bounded by some of row invariants.

In [5], it is proved that using the vanishing of Tor and Ext there are some restrictions on the minimal free resolutions of finitely generated A-modules as follows: Let M be a module over a local ring $(A,\mathfrak{m})$. We define $s(M) := \inf\{t \geq 1 : \mathrm{Soc}(M) \not\subset \mathfrak{m}^t M\}$, where $\mathrm{Soc}(M) := \mathrm{Hom}(A/\mathfrak{m}, M)$ denotes the socle of M. If $\mathrm{Soc}(M) \subseteq \mathfrak{m}^t M$, for all $t \geq 1$, then we set $s(M) := \infty$ and $\mathfrak{m}^\infty = \{0\}$.

Fact 1.1. ([3] Proposition 1.2) Let $(A,\mathfrak{m})$ be a local ring. Let

$$(F_\bullet,\varphi_\bullet) : \cdots \longrightarrow A^{n_{j+1}} \xrightarrow{\varphi_{j+1}} A^{n_j} \xrightarrow{\varphi_j} A^{n_{j-1}} \longrightarrow \cdots \longrightarrow A^{n_0} \longrightarrow 0$$

denote a minimal free resolution of a finitely generated A-module M. Let $i > 0$ be an integer which is less than $\mathrm{projdim}\, M$. If N is an A-module such that $\mathrm{Tor}_i^A(M,N) = 0$, then each column of φ_{i+1} contains an element outside $\mathfrak{m}^{s(N)}$.

The above fact can be stated as the following theorem:

Theorem 1.2. *Let $(A,\mathfrak{m})$ be a Noetherian local ring. Then there is an integer $t \geq 1$ such that for each finitely generated A-module M of infinite projective dimension, the ideal generated by the entries of the map φ_i is not contained in $\mathfrak{m}^t$ for all $i > 1 + \mathrm{depth}\, A$ ($i > \mathrm{depth}\, A$ if A is Cohen-Macaulay), where*

$$(F_\bullet,\varphi_\bullet) : \cdots \longrightarrow A^{n_{j+1}} \xrightarrow{\varphi_{j+1}} A^{n_j} \xrightarrow{\varphi_j} A^{n_{j-1}} \longrightarrow \cdots \longrightarrow A^{n_0} \longrightarrow 0$$

is a minimal (free) resolution of M.

The above fact and theorem give us the following homological invariant: We denote by $\varphi_i(M)$ the i-th map in a minimal resolution of a finitely generated A-module M.

Definition 1.3. Let $(A,\mathfrak{m})$ be a Noetherian local ring. We define $col(A) = \inf\ \{t \geq 1$: for each finitely generated A-module M of infinite projective dimension, each column of $\varphi_i(M)$ contains an element outside $\mathfrak{m}^t$, for all $i > 1 + \mathrm{depth}\ A$ $\}$.

Auslander Index is defined by using a δ-invariant in [2,3]. For a finite module M on a Cohen-Macaulay local ring A of dimension d, $\delta(M)$ is defined to be 0 if M is a homomorphic image of a maximal

Cohen-Macaulay module without free summand. On a Cohen-Macaulay local ring A, $Index(A)$ is defined to be the smallest integer n such that $\delta(A/\mathfrak{m}^n) \neq 0$ for a Cohen-Macaulay local ring $(A, \mathfrak{m})$. (On a Gorenstein local ring, it is the smallest integer n such that there is an epimorphism $X \coprod A^n \to M$ with a maximal Cohen-Macaulay module X with no free summand.) We note here that $Index(A)$ is not necessarily finite ([3]) while $col(A)$ is always finite.

2. Main theorems

When a Cohen-Macaulay local ring A has a canonical module, Ding found ([3]) the necessary and sufficient condition for the finiteness of index: $Index(A) < \infty$ if and only if A is Gorenstein on the punctured spectrum, i.e., $A_{\mathfrak{p}}$ is Gorenstein for all nonmaximal prime ideal $\mathfrak{p}$.

Leuschke proved [11] the following theorem when a ring has a Gorenstein module which is a maximal Cohen-Macaulay module of finite injective dimension: a canonical module is a Gorenstein module.

Theorem 2.1. ([11]) *Let A be a Cohen-Macaulay local ring of dimension d. Assume that A has a Gorenstein module G. Then $Index(A) < \infty$ if and only if A is Gorenstein on the punctured spectrum if and only if every maximal Cohen-Macaulay module is a d-th syzygy.*

It is easy to show that if A is a Gorenstein ring, then $Index(A)$ is always finite by the previous comments, moreover, it is known ([2]) that $Index(A) \leq \ell\ell(A)$, where $\ell\ell(A)$ is a generalized Loewy length. When does $Index(A)$ have a finite value without the assumption of the existence of a Gorenstein module? We partly answer to this question by using Column invariant.

It is quite interesting that there is a relation between Auslander Index and Column invariant because they are defined from a quite different circumstances: Index is induced from the Cohen-Macaulay approximation, and Column invariant is induced from the entries of minimal free resolutions of modules.

First, we modify Index(A) as follows:

Definition 2.2. Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d. For a finite A-module M, $\delta_0(M)$ is defined to be 0 if there is an epimorphism $X \to M$ for a $(d+1)$-th syzygy X in a minimal free resolution of a finite A-module. Otherwise, we write $\delta_0(M) \neq 0$. We define

$$Index_0(A) = \inf\{t : \delta_0(A/\mathfrak{m}^t) \neq 0\}.$$

One of the applications of Column invariant is the following well-known fact.

FACT 2.3. ([4, 5]) Let A be a Noetherian local ring, and M a finitely generated A-module of infinite projective dimension. Then the i-th syzygy of M has no free summand for $i > depth(A)$.

Since a $(d+1)$-th syzygy in a minimal free resolution is a maximal Cohen-Macaulay module, and has no free summands by the above fact, if $\delta_0(M) = 0$, then $\delta(M) = 0$. More generally, we have

$$Index_0(A) \leq Index(A).$$

Now, we show that $Index_0(A)$ is always finite, while $Index(A)$ is not. We also show if every maximal Cohen-Macaulay module is a $(d+1)$-st syzygy, then $Index(A) < \infty$ without the assumption of the existence of a Gorenstein module.

THEOREM 2.4. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d. Then*

(1) $Index_0(A) = col(A)$. In particular, $Index_0(A)$ is finite.

(2) If every maximal Cohen-Macaulay module without free summand is a $(d+1)$-th syzygy, then $Index_0(A) = Index(A)$. In particular, $Index(A) < \infty$.

Proof. (1) It can be proved with the same method in the proof of the theorem in [8]. However, we sketch the proof for reader's convenience: By the definition of $col(A)$, there exists a finite A-module N such that for a minimal resolution of N,

$$\cdots \to A^{n_i} \xrightarrow{\varphi_i} A^{n_{i-1}} \to \cdots \to A^{n_1} \xrightarrow{\varphi_1} A^{n_0} \to N \to 0$$

each column of $\varphi_i \not\subseteq \mathfrak{m}^{col(A)}$, but some column of $\varphi_i \subseteq \mathfrak{m}^{col(A)-1}$ for some $i \geq \text{depth}(A) + 2$. We may assume that the first column of φ_i is contained in $\mathfrak{m}^{col(A)-1}$ by a change of basis. We denote $\text{Im}(\varphi_{i-1})$ by $\text{Syz}_{i-1}(N)$, i.e., the $(i-1)$-th syzygy of N. Then we can show that there exists an epimorphism $\text{Syz}_{i-1}(N) \longrightarrow A/\mathfrak{m}^{col(A)-1}$. Thus by the definition of δ_0-invariant, $\delta_0(A/\mathfrak{m}^{col(A)-1}) = 0$ since $i - 1 \geq d + 1$ and so $\text{Syz}_{i-1}(N)$ is a $(d+1)$-th syzygy. Hence we have $col(A) \leq Index_0(A)$.

Conversely, if we let $Index_0(A) = n$, then we have $\delta_0(A/\mathfrak{m}^{n-1}) = 0$. Thus there exists a $(d+1)$-th syzygy X in a minimal resolution such that $\epsilon : X \to A/\mathfrak{m}^{n-1} \to 0$ is an epimorphism. Now let

$$G_\bullet \ : \ \cdots \longrightarrow A^{m_1} \xrightarrow{\psi} A^{m_0} \xrightarrow{\phi} X \longrightarrow 0$$

be a minimal free resolution of X. Then we can show that every entry of the first column of ψ belongs to $\mathfrak{m}^{n-1}$. Since X is a $(d+1)$-th syzygy in a minimal free resolution of some finite A-module M, by considering the following minimal resolution of M,

$$\begin{array}{ccccccc} \cdots \to A^{m_1} \xrightarrow{\psi} & A^{m_0} & \longrightarrow & A^{n_d} \to \cdots \to A^{n_0} \to M \to 0 \\ & & \searrow \quad \nearrow & \\ & & X & \end{array}$$

we can conclude that $col(A) \geq Index_0(A)$. We know that $col(A)$ is always finite, and so $Index_0(A)$ is also finite.

For (2), let $Index_0(A) = t$. Suppose to the contrary $\delta(A/\mathfrak{m}^t) = 0$. Then there is an epimorphism $X \to A/\mathfrak{m}^t$ for a maximal Cohen-Macaulay module with no free summand X. By assumption, X is a $(d+1)$-th syzygy, and so $\delta_0(A/\mathfrak{m}^t) = 0$, which is a contradiction. Thus $\delta(A/\mathfrak{m}^t) \neq 0$, and so $Index(A) \leq t = Index_0(A)$. Hence $Index(A) < \infty$. □

If A is a Gorenstein local ring, then it is known ([2]) $Index(A)$ is always finite (in fact, A is itself a canonical module). I don't know how strong the assumption is in Theorem 2.4 (2). However, it is expected from the following theorem that the assumption above would not imply for A to be Gorenstein.

THEOREM 2.5. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d. If A is Gorenstein, then any maximal Cohen-Macaulay A-module without free summands is an ℓ-th syzygy for any integer $\ell \geq 0$. Moreover, if A has a canonical module ω_A, then the converse also holds. In fact, if every maximal Cohen-Macaulay A-module without free summands is a $(d+1)$-th syzygy, then A is Gorenstein.*

Proof. The first part was proved in [7]. For the proof of the second part, suppose that A has a canonical module ω_A. If ω_A has a free summand, then clearly A is Gorenstein. Since ω_A is a maximal Cohen-Macaulay module, ω_A is an ℓ-th syzygy for any integer $\ell \geq 0$ by assumption. If we choose $\ell = d + 1$, then we may have a short exact sequence

$$(*) \qquad\qquad 0 \to \omega_A \xrightarrow{\alpha} F \to X \to 0,$$

where F is a free module, and X is a d-th syzygy. By Depth Lemma, we know that X is a maximal Cohen-Macaulay module. Then we have a long exact sequence

$$0 \to \mathrm{Hom}_A(X, \omega_A) \to \mathrm{Hom}_A(F, \omega_A) \to \mathrm{Hom}_A(\omega_A, \omega_A) \\ \to \mathrm{Ext}^1_A(X, \omega_A) \to \cdots .$$

Since $\mathrm{Ext}^1_A(X, \omega_A) = 0$ by [1], we know that for $id \in \mathrm{Hom}_A(\omega_A, \omega_A)$ there is $h \in \mathrm{Hom}_A(F, \omega_A)$ such that $h \circ \alpha = id$. Thus $(*)$ is split, i.e., ω_A is a direct summand of F, and so $\omega_A \cong A$. Hence A is Gorenstein. □

From the above theorem and Theorem 2.4, we have the following well-known fact:

Corollary 2.6. *Let $(A, \mathfrak{m})$ be a Gorenstein local ring. Then $Index(A)$ is finite.*

For a Cohen-Macaulay local ring A, we define $row(A) = \inf \{t \geq 1$: for each finitely generated A-module M of infinite projective dimension, each row of $\varphi_i(M)$ contains an element outside $\mathfrak{m}^t$, for all $i >$ depth A $\}$. Then we have the following corollary:

Corollary 2.7. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d. Suppose that A has a canonical module and every maximal Cohen-Macaulay A-module without free summands is a $(d+1)$-th syzygy. Then*

$$col(A) = row(A).$$

Proof. By the above theorem, A is Gorenstein, and so the conclusion follows from Proposition 4.1 in [6]. □

We now make more comments on other numerical invariants. We recall the definitions of some invariants related to columns and rows of maps of minimal free resolutions: We define $col_{CM}(M)$ to be the smallest positive integer c such that each row of the presenting matrix of M contains an element outside $\mathfrak{m}^c$. If A is a Cohen-Macaulay local ring, we define

$col_M(A) =: \sup\{col_{CM}(M) : M$ is a maximal Cohen-Macaulay module without free summands $\}$.

$col_j(A) =: \inf\{t \geq 1 :$ for each j-th syzygy module without free summands of infinite projective dimension, each row of its presenting matrix contains an element outside $\mathfrak{m}^t\}$.

We can define $row_M(A)$ and $row_j(A)$ replacing columns by rows in a similar manner. It is interesting that $col_{CM}(A)$ is not necessarily finite, while $row_{CM}(A)$ is always finite ([10]).

REMARK 2.8. It was proved ([10]) that when A is a Cohen-Macaulay local ring of dimension d, $row_d(A) = row_{CM}(A)$, and $col_d(A) = col_{CM}(A)$, provided that $col_{CM}(A)$ is finite. If we replace the assumption by every maximal Cohen-Macaulay module without free summand being a $(d+1)$-st syzygy, the proofs become easier: (Proof) By Theorem 2.4, we can show that $col_{CM}(A) < \infty$, and now let $col_{CM}(A) = t$. Then there is a maximal Cohen-Macaulay module M without free summand such that some column of the presenting matrix of M is contained in $\mathfrak{m}^{t-1}$. Since M is a maximal Cohen-Macaulay module, we have the following minimal resolution of an A-module N such that M is a d-th syzygy of N:

$$\cdots \to A^{n_{d+1}} \xrightarrow{\phi_{d+1}} A^{n_d} \xrightarrow{\phi_d} A^{n_{d-1}} \xrightarrow{\phi_{d-1}} A^{n_{d-2}} \cdots \to A^{n_0} \to N \to 0.$$
$$\searrow \quad \nearrow$$
$$M$$

Since some column of a map φ_{d+1} is contained in $\mathfrak{m}^{t-1}$, and φ_{d+1} is a presenting matrix of d-th syzygy of N, we have $col_{CM}(A) = t \leq col_d(A)$. $row_{CM}(A) \leq row_d(A)$ can be proved in the exact same way as above.

There is a question, which is still unsolved, that the value of column invariant of a ring is less than or equal to that of the row invariant of a ring, $col(A) \leq row(A)$. We still don't know the answer of the above question, but we prove that $col_(A) \leq row_{d-2}(A)$ if A is Gorenstein on the punctured spectrum.

THEOREM 2.9. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d. Suppose that A is Gorenstein on the punctured spectrum. Then we have*

$$col(A) \leq row_{d-2}(A).$$

Proof. Let $col(A) = t$. Then there is a finitely generated A-module M of infinite projective dimension such that for some $j > 1+d$, every entry of some column of ϕ_j in $\mathfrak{m}^{t-1}$, where $(G_\bullet, \phi_\bullet)$ is a minimal resolution of N. Note that $(j-1)$-th syzygy module, say M, is a maximal Cohen-Macaulay module without free summands. Let $(F_\bullet, \varphi_\bullet)$ be a minimal free resolution of M:

$$(F_\bullet, \varphi_\bullet) : \cdots \longrightarrow A^{n_i} \xrightarrow{\varphi_i} A^{n_{i-1}} \longrightarrow \cdots \longrightarrow A^{n_1} \xrightarrow{\varphi_1} A^{n_0} \longrightarrow 0.$$

Then some column of $\varphi_1 (= \phi_j)$ is contained in $\mathfrak{m}^{t-1}$. Now, we dualize to obtain the following:

$$(F^*_\bullet, \varphi^T_\bullet) : 0 \longrightarrow \mathrm{Hom_A(M, A)} \longrightarrow A^{n_0} \xrightarrow{\varphi_1^T} A^{n_1} \longrightarrow \cdots$$
$$\longrightarrow A^{n_{d-1}} \longrightarrow D \longrightarrow 0,$$

where φ_i^T is a transpose matrix of φ_i, and $D = \text{coker}(\varphi_{d-1}^T)$. Since $A_\mathfrak{p}$ is Gorenstein for all $\mathfrak{p} \neq \mathfrak{m}$, we can show that $(F_\bullet^*, \varphi_\bullet^T)$ is exact by using the method in [11]. We note that φ_1^T is the presenting matrix of $(d-2)$-th syzygy of D, and some row of φ_1^T is contained in $\mathfrak{m}^{t-1}$. Thus we have $col(A) = t \leq row_{d-2}(A)$. □

We close this section by putting forth some of questions.

QUESTION 2.10. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Is $col(A)$ the same as $col(A/\mathfrak{m})$? If A is Cohen-Macaulay, then*

$$row(A) = row(A/\mathfrak{m})?$$

For a finitely generated A-module M, we define $\text{col}(M)$ [resp. $\text{row}(M)$] to be 1 if $\text{projdim}\, M < \infty$. Otherwise, we define $\text{col}(M)$ [resp. $\text{row}(M)$] to be the smallest c such that for each $i > 1+\text{depth}\, A$ [resp. $i > \text{depth}\, A$], each column [resp. row] of φ_i contains an element outside $\mathfrak{m}^c$, where $(F_\bullet, \varphi_\bullet)$ is a minimal resolution of M.

Using a mapping cone of complexes, it can be shown that $col(A)$ [resp. $row(A)$] has the same value as $col(M)$ [resp. $row(M)$] for some finitely generated A-module M of depth 0.

QUESTION 2.11. *Let A be a Cohen-Macaulay local ring. Then*

$$col(A) \leq row(A)?$$

If A is a Gorenstein local ring, then it is known ([6]) that $col(A) = row(A)$. If a ring is not Gorenstein, then the equality does not necessarily hold. For example, if $R = k[\![t^e, t^{e+1}, t^{(e-1)e-1}]\!]$, where k is a field and $e \geq 4$, then R is a Cohen-Macaulay local ring of dimension 1, but not Gorenstein. It is known ([9]) that $col(R) = 2 < e-1 = row(R)$.

QUESTION 2.12. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d.*

(1) If every maximal Cohen-Macaulay A-module without free summands is an ℓ-th syzygy for any integer $\ell \geq 0$, then is A Gorenstein?

(2) If every maximal Cohen-Macaulay A-module without free summands is an $(d+1)$-th syzygy, then is A Gorenstein?

QUESTION 2.13. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d, which does not have necessarily a Gorenstein module. If every maximal Cohen-Macaulay A-module without free summands is an d-th syzygy, then is $Index(A)$ finite?*

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