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THE DOMINATION COVER PEBBLING NUMBER OF SOME GRAPHS

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ABSTRACT. A *pebbling move* on a connected graph G is taking two pebbles off of one vertex and placing one of them on an adjacent vertex. The *domination cover pebbling number* $\psi(G)$ is the minimum number of pebbles required so that any initial configuration of pebbles can be transformed by a sequence of pebbling moves so that the set of vertices that contain pebbles forms a domination set of G. We determine the *domination cover pebbling number* for fans, fuses, and pseudo-star.

1. Introduction

Since Chung[1] introduced the concept of pebbling in graph theory, several researchers including Lagarias, Saks and Hurlbert made progress in the study of pebbling in graph theory.

Throughout this paper G will denote a simple connected graph. Consider a connected graph with a fixed number of pebbles distributed on its vertices. A *pebbling move(step)* consists of removing two pebbles from one vertex u and then placing one pebble at an adjacent vertex v. We say that we can pebble to a vertex v, the target(root) vertex, if we can apply pebbling moves repeatedly so that it is possible to reach a configuration with at least one pebble at v. We define the *pebbling number of a vertex* v for a graph G, denoted by $f(G, v)$, to be the smallest integer m which guarantees that any starting pebble configuration with m pebbles allows pebbling to v. We define the *pebbling number of* G, denoted by $f(G)$ as the maximum of $f(G, v)$, over all vertices v.

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The *cover pebbling number* $\gamma(G)$ is defined to be the minimum number of pebbles needed to place a pebble on every vertex of the graph using a sequence of pebbling moves, regardless of the initial configuration by Crull et al. [2]. A set S of some vertices in G is a *dominating set* if every vertex in G is either in S or adjacent to some element in S. Gardner et al. [4] combine these two ideas, *cover pebbling* and *domination*, to introduce a new graph invariant called the *domination cover pebbling number*, $\psi(G)$, of a graph. The *domination cover pebbling number* $\psi(G)$ of a graph G is the minimum number of pebbles that must be placed on $V(G)$ such that after a sequence of pebbling moves, the set of vertices with pebbles forms a dominating set of G, regardless of the initial configuration of pebbles. The motivation of Gardner et al. for this definition comes from a hypothetical situation in which one wishes to transport monitors along the edges of a network that could ultimately "watch" each vertex-but half the devices are lost during each move. The pebbles may be placed on any of the vertices of G, and S depends, in general, on the initial configuration - most importantly, however, S need not equal a minimum dominating set.

Gardner et al.[4] calculated the domination cover pebbling number for paths, cycles and complete binary trees.

In this paper, we determine the domination cover pebbling number of fans, fuses and pseudo-stars. For a finite set S, $|S|$ denotes the number of elements in S. For any configuration C on G and a vertex v of G, we denote by $C(v)$ the number of pebbles on v in the configuration C.

2. Domination Cover Pebbling for Fans

Gardner et al.[4] determined the domination cover pebbling number for some family of graphs.

THEOREM 2.1 ([4]). *For the complete graph K_n on n vertices, $\psi(K_n) = 1$.*

The wheel graph, denoted by W_n, is the graph with $V(W_n) = \{h, v_1, \cdots, v_n\}$, where h is called the hub of W_n, and $E(W_n) = C_n \bigcup \{hv_1, hv_2, ..., hv_n\}$, where C_n denotes the cycle graph on n vertices $v_1, v_2, ..., v_n$.

THEOREM 2.2 ([4]). *For the wheel graph W_n , $\psi(W_n) = n - 2$, for $n \geq 3$.*

First, we find the domination cover pebbling number of the star graph S_n with n vertices.

THEOREM 2.3. *For the star graph S_n, $\psi(S_n) = n - 1$.*

This result is obvious.

A *fan graph*, denoted by F_n, is a path P_n plus an extra vertex connected to all vertices of the path P_n.

Throughout this paper, a fan graph with vertices $v_0, v_1, \cdots, v_n$ in order means the fan graph F_n whose vertices of the path P_n are $v_1, \cdots, v_n$ in order and whose extra vertex is v_0.

THEOREM 2.4. *For the fan graph, $\psi(F_n) = n - 1$, $n \geq 3$.*

Proof. $\psi(F_n) > n - 2$ because placing one pebble on each of $n - 2$ consecutive vertices $v_1, \cdots, v_{n-2}$ on P_n leaves the vertex v_n of F_n non-dominated. If there is a pair of pebbles on any vertex, move it to the center v_0, then the domination is complete. Likewise, if there is a pebble at v_0, F_n is dominated. Thus, consider all configurations such that each pebbled vertex contains just one pebble. If there are $n - 1$ pebbled vertices, then there are just two non-pebbled vertices. It is easy to see that these two non-pebbled vertices are dominated. Therefore, $\psi(F_n) = n - 1$. □

The following family of graphs which was introduced by Watson[5]. The pseudo-star graph, denoted by $H_w(n)$, is defined to be a star graph of order $n + 1$ with w consecutive additional edges added to make the graph induced by one subset of $w + 1$ outer vertices connected.

THEOREM 2.5. *For the pseudo-star graph, $\psi(H_w(n))= n$, $1 \leq w \leq n - 2$,*

Proof. Let $V(H_w(n))$ be $\{h, v_1, \cdots, v_n\}$ and $E(H_w(n)))$ $\{hv_1, hv_2, \cdots, hv_n, v_1v_2,\ v_2v_3, \cdots, v_wv_{w+1}\}$. First, $\psi(H_w(n)) > n - 1$ because placing one pebble on each of $(n - 1)$ consecutive vertices

$v_1, \cdots, v_{n-1}$ leaves the vertex v_n non-dominated. If there is a pair of pebbles on any vertex, move it to the center, then the domination is complete. Likewise, if there is a pebble at h, $H_w(n)$ is dominated. Thus, consider all configurations such that each pebbled vertex contains just one pebble. If there are n pebbled vertices, then there is just one non-pebbled vertex. It is easy to see that this non-pebbled vertex is dominated. Therefore, $\psi(H_w(n)) = n$. □

3. Domination Cover Pebbling for fuses

The class of *fuses* is defined as follows. The vertices of a fuse $F_\ell(k)$ ($\ell \geq 2$ and $k \geq 2$) are $v_1, \cdots, v_n$ with $n = \ell + k$, so that the first ℓ vertices form a path from v_1 to v_ℓ, and the remaining vertices $v_{l+1}, \cdots, v_n$ are independent and adjacent only to v_ℓ. The path is sometimes called the *wick*, while the remaining vertices are sometimes called the *sparks*. For example, $F_2(k)$ is the star S_{k+2} on $k+2$ vertices. The fact that $\psi(S_n) = n - 1$ serves as the base case for the following result.

THEOREM 3.1. *For the fuse graph,*

(1)
$$\psi\big(\,F_\ell\,(n)\,\big) = \begin{cases} \dfrac{2^{\ell+2} - 2^{\alpha}}{7} + (k-1)\,, & \textit{if} \quad \ell - 1 \equiv \alpha \neq 0 \pmod 3 \\[2ex] \dfrac{2^{\ell+2} - 2^{3}}{7} + (k-1)\,, & \textit{if} \quad \ell - 1 \equiv 0 \pmod 3 \end{cases}$$

Proof. Induction on l shows that so many pebbles suffice to dominate the fuse. Regarding the base case $l = 2$, we point out that $F_2(k)$ is the star on $k+2$ vertices. Consider the configuration D such that $D(v_i)$=1 for $i = l+2, \cdots, n$, $D(v_j)$=0 for $j = 1, \cdots, \ell$, and $D(v_{\ell+1}) = \frac{8 \cdot 2^{\ell-1} - 2^{\alpha}}{7}$. We need at least $2^{\ell-1} + 2^{\ell-4} + \cdots + 2^{\alpha}$ ($0 \neq \alpha \equiv \ell - 1 \pmod 3$ and $\alpha = 3$ if $\ell - 1 \equiv 0 \pmod 3$) pebbles to dominate $\{v_1, \cdots, v_{\ell-1}\}$. $v_{\ell+2}$

dominates v_ℓ. But

$$2^{\ell-1} + 2^{\ell-4} + \cdots + 2^{\alpha} = \frac{8 \cdot 2^{\ell-1} - 2^{\alpha}}{7}$$

Thus, under this configuration D,

$$\psi\big(F_\ell\,(k)\big) \geq \frac{8 \cdot 2^{\ell-1} - 2^{\alpha}}{7} + (k-1)\,.$$

We now use induction on ℓ to show that $\psi\big(F_\ell\,(k)\big) \leq (1)$. The assertion is clear for $\ell = 2$. Therefore, we assume it is true for all s, when $2 \leq s \leq \ell - 1$. Consider an arbitrary configuration of $F_\ell\,(k)$ having (1) pebbles. Clearly we can cover dominate $\{v_1,\ v_2,\ v_3\ \}$ is a finite number of moves with $2^{\ell-1}$ pebbles or less. Thus, we need to dominate $F_{\ell-3}\,(k)$ with the remaining

$$(1) - 2^{\ell-1} = \frac{2^{(\ell-3)+2} - 2^{\alpha}}{7} + (k-1)$$

pebbles. This number of pebbles is enough to dominate $F_{\ell-3}\,(k)$ by hypothesis. Thus,

$$\psi\big(F_\ell\,(k)\big) \geq (1)\,,$$

completing the proof. □

Determination of the ψ values for several other families of graphs is an still open question.

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