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A SHARP RESULT FOR A NONLINEAR LAPLACIAN DIFFERENTIAL EQUATION

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Abstract. We investigate relations between multiplicity of solutions and source terms in a elliptic equation. We have a concerne with a sharp result for multiplicity of a nonlinear Laplacian differential equation.

1. Introduction

A semilinear elliptic boundary value problem under the Dirichlet boundary condition

$$
\begin{aligned} Au + bu^+ - au^- &= t_1\phi_1 + t_2\phi_2 \quad \text{in} \quad \Omega. \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{1.1}
$$

Here, the second order elliptic differential operator

$$
A = \sum_{1\le i,j\le n} a_{i,j}(x) D_i D_j
$$

is a mapping from $L^2(\Omega)$ into itself with compact inverse, with eigenvalues $-\lambda_i$, each repeated as often as multiplicity, where $a_{ij} = a_{ji} \in C^\infty(\bar{\Omega})$.

Ω be a bounded set in $\mathbf{R}^n (n \ge 1)$ with smooth boundary $\partial\Omega$. We denote ϕ_n to be the eigenfuction corresponding to $\lambda_n (n = 1, 2, \cdots)$ and the eigenfuction such that $\phi_1 > 0$ in Ω and $\int_\Omega \phi_1^2 = 1$. We will also let ϕ_i denote the eigeneunctions corresponding to λ_i normalized by inner product

$$
(\phi_i, \phi_j) = \int_\Omega \phi_i\phi_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}
$$

and the set $\{\phi_n |\ n = 1, 2, \cdots\}$ is an orthogonal set in Hilbert space H.

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We suppose that $a < \lambda_1, \lambda_2 < b < \lambda_3$. we have a concern with the multiplicity of solutions of (1.1) when $h = t_1\phi_1 + t_2\phi_2$ is generated by two eigenfunctions ϕ_1 and ϕ_2. Then equation (1.1) is equivalent to

$$Au + bu^+ - au^- = h \quad \text{in} \quad H. \tag{1.2}$$

Hence we will study the equation (1.2). We use the contraction mapping principle to reduce the problem from an infinite dimensional space in H to a finite dimensional one.

Let V be the two dimensional subspace of H spanned by $\{\phi_1, \phi_2\}$ and W be the orthogonal complement of V in H. Let P be an orthogonal projection H onto V. Then every element $u \in H$ is expressed as

$$u = v + w,$$

where $v = Pu, w = (I - P)u$. Hence equation (1.2) is equavelent to a system

$$Aw + (I - P)(b(v + w)^+ - a(v + w)^-) = 0 \tag{1.3}$$

$$Av + P(b(v + w)^+ - a(v + w)^-) = t_1\phi_1 + t_2\phi_2. \tag{1.4}$$

Here we look on (1.3) and (1.4) as a system of two equation in the two unknows v and w.

We know in [2] that *for fixed $v \in V$ (1.3) has a unique solution $w = \theta(v)$. Furthermore, $\theta(v)$ is Lipschitz continuous(with respect to the L^2-norm) in terms of v.*

Hence, the study of the multipicity of solution of (1.2) is reduced to the study of the multipicity of solutions of an equivalent problem

$$Av + P(b(v + \theta(v))^+ - a(v + \theta(v))^-) = t_1\phi_1 + t_2\phi_2 \tag{1.5}$$

defined on the two dimensional subspace V spanned by $\{\phi_1, \phi_2\}$.

While one feels intuively that (1.5) ought to be easier to solve than (1.2), there is the disadvantage of an implicitly defined term $\theta(v)$ in the equation. However, in our case, it turns out that we know $\theta(v)$ for some special $v's$.

If $v \geq 0$ or $v \leq 0$, then $\theta(v) \equiv 0$. For example, let us take $v \geq 0$ and $\theta(v) = 0$. Then equation (1.3) reduces to

$$A0 + (I - P)(bv^+ - av^-) = 0,$$

which is satisfied because $v^+ = v, v^- = 0$ and $(I - P)v = 0$, since $v \in V$. Since the subspace V is spanned by $\{\phi_1, \phi_2\}$ and ϕ_1 is a positive eigenfuction, there exists a cone C_1 defined by

$$C_1 = \{v = c_1\phi_1 + c_2\phi_2 \mid c_1 \geq 0, |c_2| \leq qc_1\}$$

for some $q > 0$ so that $v \geq 0$ for all $v \in C_1$ and a cone C_3 defined by

$$C_3 = \{v = c_1\phi_1 + c_2\phi_2 \mid c_1 \leq 0, |c_2| \leq q|c_1|\}$$

so that $v \leq 0$ for all $v \in C_3$.

Thus, even if we do not know $\theta(v)$ for all $v \in V$, we know $\theta(v) \equiv 0$ for $v \in C_1 \cup C_3$. Now we define a map $\Pi : V \to V$ given by

$$\Pi(v) = Av + P(b(v + \theta(v))^+ - a(v + \theta(v))^-), \quad v \in V. \tag{1.6}$$

Π of (1.6) is continuous on V, and we can see that for $v \in V$

$$\Pi(cv) = c\Pi(v) \quad (c \geq 0). \tag{1.7}$$

We investigate the image of the cones C_1, C_3 under Π. First, we consider the image of cone C_1. If $v = c_1\phi_1 + c_2\phi_2 \geq 0$, we have

$$\begin{aligned}\Pi(v) &= Av + P(b(v + \theta(v))^+ - a(v + \theta(v))^-)\\ &= -c_1\lambda_1\phi_1 - c_2\lambda_2\phi_2 + b(c_1\phi_1 + c_2\phi_2)\\ &= c_1(b - \lambda_1)\phi_1 + c_2(b - \lambda_2)\phi_2.\end{aligned}$$

Thus the image of the rays $c_1\phi_1 \pm qc_1\phi_2 (c_1 \geq 0)$ can be caculated and they are

$$c_1(b - \lambda_1)\phi_1 \pm qc_1(b - \lambda_2)\phi_2 \quad (c_1 \geq 0). \tag{1.8}$$

Therefore if $a < \lambda_1, \lambda_2 < b < \lambda_3$, then Π maps C_1 onto the cone

$$D_1 = \left\{ d_1\phi_1 + d_2\phi_2 \;\middle|\; d_1 \geq 0, |d_2| \leq q\left(\frac{b-\lambda_2}{b-\lambda_1}\right) d_1 \right\}.$$

Second, we consider the image of the cone C_3. If

$$v = -c_1\phi_1 + c_2\phi_2 \leq 0 \quad (c_1 \geq 0, |c_2| \leq qc_1),$$

we have

$$\begin{aligned} \Pi(v) &= Av + P(b(v+\theta(v))^+ - a(v+\theta(v))^-) \\ &= Av + P(av) \\ &= c_1(\lambda_1 - a)\phi_1 - c_2(\lambda_2 - a)\phi_2. \end{aligned}$$

Thus the image of the rays $-c_1\phi_1 \pm qc_1\phi_2 (c_1 \geq 0)$ can be caculated and they are

$$c_1(\lambda_1 - a)\phi_1 \pm qc_1(\lambda_2 - a)\phi_2 \quad (c_1 \geq 0). \tag{1.9}$$

Therefore, if $a < \lambda_1, \lambda_2 < b < \lambda_3$, then Π maps C_3 onto the cone

$$D_3 = \left\{ d_1\phi_1 + d_2\phi_2 \;\middle|\; d_1 \geq 0, |d_2| \leq q\left(\frac{\lambda_2 - a}{\lambda_1 - a}\right) d_1 \right\}.$$

2. The existence of solutions

We note that $D_1 \subset D_3$ since $a < \lambda_1, \lambda_2 < b < \lambda_3$. We investigate the images of the cones C_2, C_4 under Π. we suppose that $a < \lambda_1, \lambda_2 < b < \lambda_3, h = t_1\phi_1 + t_2\phi_2$. Now we set

$$C_2 = \{v = c_1\phi_1 + c_2\phi_2 \;\mid\; c_2 \geq 0, c_2 \geq q|c_1|\},$$

$$C_4 = \{v = c_1\phi_1 + c_2\phi_2 \;\mid\; c_2 \leq 0, |c_2| \geq q|c_1|\},$$

Then the union of C_1, C_2, and C_3, C_4 are the space V.

We note that $D_1 \subset D_3$ since $a < \lambda_1, \lambda_2 < b < \lambda_3$. We investigate the images of the cones C_2, C_4 under Π.

To investigate the images of the cones C_2, C_4, we need the following lemma.

Lemma 2.1. *There exists $d > 0$ so that*

$$(\Pi(v), \phi_1) \geq d|c_2| \quad \text{for all} \quad v = c_1\phi_1 + c_2\phi_2 \in V.$$

Proof. Let $g(u) = bu^+ - au^-$ and let $v = c_1\phi_1 + c_2\phi_2$. Let $u = c_1\phi_1 + c_2\phi_2 + \theta(c_1, c_2)$. Then

$$\Pi(c_1\phi_1 + c_2\phi_2) = A(c_1\phi_1 + c_2\phi_2) + P(g(c_1\phi_1 + c_2\phi_2 + \theta(c_1, c_2))).$$

So we have

$$(\Pi(v), \phi_1) = ((A + \lambda_1)(c_1\phi_1 + c_2\phi_2), \phi_1) + (g(u) - \lambda_1 u, \phi_1).$$

The first term is zero because $(A + \lambda_1)\phi_1 = 0$ and A is a self-adjoint. The second term satisfies

$$\begin{aligned} g(u) - \lambda_1 u &= bu^+ - au^- - \lambda_1(u^+ - u^-) \\ &= (b - \lambda_1)u^+ + (\lambda_1 - a)u^- \geq \gamma|u|, \end{aligned}$$

where $\gamma = \min\{b - \lambda_1, \lambda_1 - a\} > 0$. Therefore

$$(\Pi(v), \phi_1) \geq \gamma \int |u|\phi_1.$$

Now there exists $d > 0$ so that $\gamma\phi_1 \geq d|\phi_2|$ and therefore

$$\gamma \int |u|\phi_1 \geq d \int |u||\phi_2| \geq d\left| \int u\phi_2 \right| = d|c_2|. \qquad \square$$

Lemma 2.1 means that the image of Π is contained in the right half-plane. That is, $\Pi(C_2)$ and $\Pi(C_4)$ are the cones in the right half-plane. The image of C_2 is the cone containing

$$D_2 = \left\{ d_1\phi_1 + d_2\phi_2 \;\middle|\; d_1 \geq 0, -q\left(\frac{\lambda_2 - a}{\lambda_1 - a}\right)d_1 \leq d_2 \leq q\left(\frac{\lambda_2 - b}{\lambda_1 - b}\right)d_1 \right\},$$

and the image of C_4 under Π is the containing

$$D_4 = \left\{ d_1\phi_1 + d_2\phi_2 \;\middle|\; d_1 \geq 0, -q\left(\frac{\lambda_2 - b}{\lambda_1 - b}\right)d_1 \leq d_2 \leq q\left(\frac{\lambda_2 - a}{\lambda_1 - a}\right)d_1 \right\}.$$

We consider the restriction $\Pi|_{C_i}(1 \leq i \leq 4)$ of Π to the cone C_i. Let $\Pi_i = \Pi|_{C_i}$, i.e.,

$$\Pi_i : C_i \to V.$$

We consider the segments s_2 and s_4 as follows

$$s_2 = \left\{ \phi_1 + d_2\phi_2 \ \middle|\ -q\left(\frac{\lambda_2 - a}{\lambda_1 - a}\right) \leq d_2 \leq q\left(\frac{\lambda_2 - b}{\lambda_1 - b}\right) \right\},$$

$$s_4 = \left\{ \phi_1 + d_2\phi_2 \ \middle|\ -q\left(\frac{\lambda_2 - b}{\lambda_1 - b}\right) \leq d_2 \leq q\left(\frac{\lambda_2 - a}{\lambda_1 - a}\right) \right\}.$$

We investigate the inverse image $\Pi_2^{-1}(s_2), \Pi_4^{-1}(s_4)$. We note that $\Pi_i(C_i)(i = 2, 4)$ contains D_i.

By (1.7) and Lemma 2.1, we can see the following lemma.

Lemma 2.2. *Let $\sigma_i(i = 2, 4)$ be any simple path in D_i with end points on ∂D_i, where each ray (starting from the origin) in D_i intersect only one point of σ_i. Then the inverse image $\Pi_i^{-1}(\sigma_i)$ of σ_i is a simple path in C_i with end points on ∂C_i, where any ray in C_i, starting from the origin, intersects only one point of this path.*

With Lemma 2.1 and Lemma 2.2, we have the following theorem.

Theorem 2.3. *(a) The restriction $\Pi_i : C_i \to D_i(i = 1, 3)$ is bijective.*

(b) $\Pi : C_j \to D_j(j = 2, 4)$ is surjective. Therefore, Π maps V onto D_3.

Proof. First, we shall show that $\Pi_1 : C_1 \to D_1$ is bijective. By (1.8), the restriction Π_1 maps C_1 onto D_1. We consider the segment

$$s_1 = \left\{ \phi_1 + d_2\phi_2 \ \middle|\ |d_2| \leq q\left(\frac{b - \lambda_2}{b - \lambda_1}\right) \right\}.$$

Then the inverse image $\Pi_1^{-1}(s_1)$ is a segment

$$\mathcal{S}_1 = \left\{ \frac{1}{b - \lambda_1}(\phi_1 + c_2\phi_2) \ \middle|\ |c_2| \leq q \right\}.$$

By Lemma 2.2, $\Pi_1 : C_1 \to D_1$ is bijective. Second, in the same way we can show that $\Pi_3 : C_3 \to D_3$ is bijective. (b) By (1.9) and Lemma 2.2, the restriction $\Pi_j : C_j \to D_j(j = 2, 4)$ is surjective. □

We note that all cones D_2, D_3, D_4 contain the cone D_1. Also D_3, D_2 contain the cone $D_2 \backslash D_1$, and D_3, D_4 contain the cone $D_4 \backslash D_1$.

Hence we have the following theorem.

Theorem 2.3 *Suppose $a < \lambda_1, \lambda_2 < b < \lambda_3$. Let $h = t_1\phi_1 + t_2\phi_2$. Then we have the following.*

(a) If $h \in IntD_1$, then equation (1.1) has at least four solutions.

(b) If $h \in \partial D_1$, then equation (1.1) has at least three solutions.

(c) If $h \in Int(D_3 \backslash D_1)$, then equation (1.1) has at least two solutions.

(d) If $h \in \partial D_3$, then equation (1.1) has at least one solution.

(e) If h does not belong to the cone D_3, then equation (1.1) has no solution.

3. A sharp result for multiplicity

We shall investigate a sharp result for the multiplicity of equation (1.1) when the source term h belong to the interior $\text{Int}D_1$ of the cone D_1, and A is the Laplacian operator L.

Let A be a second order linear elliptic differential operator. Given a function $\eta \in L^\infty(\Omega)$, let us consider the linear eigenvalue problem

$$\begin{aligned} -Au &= \lambda\eta u \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.1}$$

An eigenvalue of (3.1) is a λ such that (3.1) has a solution $u \neq 0$. Any $\phi \neq 0$ satisfying (3.1) is an eigenfunction associated to the eigenvalue λ.

Lemma 3.1.(Comparison Property)[1]. *If $\eta \leq \xi$ in Ω, then $\lambda_k(\eta) \geq \lambda_k(\xi)$; if $\eta < \xi$ in a subset of positive measure, then $\lambda_k(\eta) > \lambda_k(\xi)$. In particular, if $\eta < \lambda_k$, then $\lambda_k(\eta) > 1$; if $\eta > \lambda_k$, then $\lambda_k(\eta) < 1$.*

Given u, we denote by $\mathcal{C}(u)$ the characteristic function of the positive set of u, that is,

$$[\mathcal{C}(u)](x) = \begin{cases} 1, & \text{if} \quad u(x) > 0 \\ 0, & \text{if} \quad u(x) \leq 0. \end{cases}$$

We set $\alpha(u) = b\mathcal{C}(u) + a\mathcal{C}(-u)$ when the measure of $\{x|u(x) = 0\}$ is zero.

Definition[6]. *We say that u is a nondegenerate solution of equation (1.1) if the problem*

$$-Av = \alpha(u)v \quad in \quad \Omega,$$
$$v = 0 \quad on \quad \partial\Omega.$$

has only the trivial solution $v \equiv 0$.

We have a concern only when A is the Laplacian operator L. We denote by K the operator $(-L)^{-1}$ from $H^{-1}(\Omega)$ into E and we consider it as a compact operator on H in a view of Sobolev's imbeding theorems.

Given $\alpha \in L^{\frac{n}{2}}(\Omega)$ one can consider the eigenvalue problem

$$\begin{aligned} -Lv &= \nu\alpha v \quad \text{in} \quad \Omega, \\ v &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.2}$$

It is well known([6]) that if $\alpha > 0$ in a set of positive measure, then the positive number ν for which (3.2) has a nontrivial solution that is a term of a sequence $\nu_1(\alpha), \nu_2(\alpha), \cdots, \nu_j(\alpha), \cdots$ diverging to $+\infty$. Since each eigenvalue ν_j has finite multiplicity, we can repeat it in the sequence as many times as its multiplicity.

We consider the nonlinear Lapacian differential equation

$$Lu + bu^+ - au^- = h(x) \quad \text{in} \quad H \tag{3.3}$$

Lemma 3.2. *Assume* $a < \lambda_1$ *and* $b \leq \lambda_k$ *for a given integer* $k > 2$. *Let* $h(x) = \phi_1 + t_2\phi_2 \in IntD_1$. *Then if u is a solution of (3.3) which changes sign in* Ω, *we have*

$$\nu_1(\alpha(u)) < 1 < \nu_{k-1}(\alpha(u)).$$

Proof. Equation (3.3) has the positive solution $u_p = (b-\lambda_1)^{-1}\phi_1 + t_2(b-\lambda_2)^{-1}\phi_2$ and a negative solution $u_n = (a-\lambda_1)^{-1}\phi_1 + t_2(a-\lambda_2)^{-1}\phi_2$. Writing (3.3) for u and u_p and substracting we get:

$$-L(u_p - u) = b(u_p - u^+) + au^-. \tag{3.4}$$

Let us use the notation

$$\hat{\alpha} = \frac{b(u_p - u^+) + au^-}{u_p - u}.$$

We have the inequalities:

$$a < \alpha(u) < \hat{\alpha} < b. \tag{3.5}$$

By (3.4), $\nu_j(\hat{\alpha}) = 1$ for some j and by (3.5) $j \in \{1, 2, \cdots, k-1\}$. We have similar computations with u_n and find a function $\check{\alpha}$ such that $\nu_j(\check{\alpha}) = 1$ for some $j' \in \{1, 2, \cdots, k-1\}$ and

$$a < \check{\alpha} < \alpha(u) < b, \tag{3.6}$$

where each inequality holds on a subset of positive measure in Ω. By Lemma 3.1, we have

$$\begin{aligned} 1 = \nu_j(\hat{\alpha}) \leq \nu_{k-1}(\hat{\alpha}) < \nu_{k-1}(\alpha(u)), \\ \nu_1(\alpha(u)) < \nu_1(\check{\alpha}) \leq \nu_{j'}(\check{\alpha}) = 1, \end{aligned}$$

which proves the lemma. □

Theorem 3.3. *Let $a < \lambda_1, \lambda_2 < b < \lambda_3$ and let $h \in IntD_1$. Then equation (3.3) has exactly four nondegenerate solutions.*

Proof. The staetment follows from Lemma 3.2 which ensures that any solution which changes sign is nondegenerate and has local degree -1. Since we know that the solutions of constant sign only are u_p and u_n and they have local degree 1, by using the equality:

$$d_{LS}(u - K(bu^+ - au^-), B(0, r), -K\phi_1) = 0$$

which is proved in [6] for large positive r. By the homotopy invariance property of degree, if $h \in \mathrm{Int}D_1$, then

$$d_{LS}(u - K(bu^+ - au^-), B(0, r), -Kh) = 0$$

for large positive r. This completes the proof. □

References

1. A. Ambrosetti and G. Prodi, *A primer of nonlinear analysis, Cambridge, University Press*, *Cambridge Studies in Advanced Math. No. 34*, 1993.
2. Q. H. Choi and K. P. Choi, *The Exitence of Solutions of an Ambrosetti-Prodi Type Nonlinear Elliptic Equation*, Far East J. Dynamical Systems, 4(1) (2002), 39-50.
3. Q. H. Choi and T. Jung, *An application of a variational reduction method to a nonlinear wave equation*, J. Differential Equations, 117 (1995), 390-410.
4. A. C. Lazer and P. J. McKenna, *On the number of solutions of a nonlinear Dirichlet problem.*, J. of Math. Analysis and Applications, 86 (1981), 282-294.
5. P. J. McKenna, *Topolgical methods for asymmetric boundary value problem, Lecture Note Series, No.11*, *Research Institute of Math., Global Analysis Research Center, Seoul National University*, 1993.
6. S. Solimini, *Some remarks on the number of solutions of some nonlinear elliptic problems*, Ann. Inst. Herni Poincare, 2 (1985), 143-156.

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