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A FIXED POINT APPROACH TO THE STABILITY OF AN ADDITIVE-QUADRATIC-QUARTIC FUNCTIONAL EQUATION

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Abstract. In this paper, we investigate the stability of a functional equation

$$f(x+3y)-5f(x+2y)+10f(x+y)-8f(x)+5f(x-y)-f(x-2y)$$
$$-2f(-x)-f(2x)+f(-2x)=0$$

by using the fixed point theory in the sense of L. Cădariu and V. Radu.

1. Introduction

The stability of functional equation has begun to become a research topic from Ulam's question [20] about the stability of group homomorphisms. Hyers [8] gave an affirmative answer to this problem for additive mappings between Banach spaces. Subsequently many mathematicians came to deal with this problem (cf. [6, 13, 18]).

In this paper, let V and W be real vector spaces and Y a real Banach space. For a given mapping $f : V \to W$, we use the following

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abbreviations

$$\begin{aligned}
f_e(x) :=& \frac{f(x)+f(-x)}{2}, \qquad f_o(x) := \frac{f(x)-f(-x)}{2},\\
Af(x,y) :=& f(x+y)-f(x)-f(y),\\
Qf(x,y) :=& f(x+y)+f(x-y)-2f(x)-2f(y),\\
Q'f(x,y) :=& f(x+2y)-4f(x+y)+6f(x)-4f(x-y)\\
&+f(x-2y)-24f(y),\\
Df(x,y) :=& f(x+3y)-5f(x+2y)+10f(x+y)-8f(x)\\
&+5f(x-y)-f(x-2y)-2f(-x)-f(2x)+f(-2x)
\end{aligned}$$

for all $x, y \in V$. Each functional equation $Af(x,y) = 0$, $Q(x,y) = 0$ and $Q'f(x,y) = 0$ is called an additive functional equation, a quadratic functional equation and a quartic functional equation, respectively. Every solution of the functional equations $Af(x,y) = 0$, $Q(x,y) = 0$ and $Q'f(x,y) = 0$ are called an additive mapping, a quadratic mapping and a quartic mapping, respectively. If a mapping can be expressed by the sum of a quartic mapping, a quadratic mapping and an additive mapping, then we call the mapping an additive-quadratic-quartic mapping. A functional equation is called an additive-quadratic-quartic functional equation provided that each solution of that equation is an additive-quadratic-quartic mapping and every additive-quadratic-quartic mapping is a solution of that equation.

Many mathematicians [7, 16, 17] investigated the stability of the additive-quadratic-quartic functional equation

$$\begin{aligned}
&f(x+2y)+f(x-2y)-2f(x+y)-2f(-x-y)-2f(x-y)\\
&-2f(y-x)+4f(-x)+2f(x)-f(2y)-f(-2y)+4f(y)+4f(-y)=0
\end{aligned}$$

for all $x, y \in V$. They proved the stability of the above functional equation by dividing into three parts: the additive part, the quadratic part and the quartic part of the given mapping f. However, in this paper, we will show the stability of another type of additive-quadratic-quartic functional equation $Df(x,y) = 0$ by using fixed point theorem without dividing into three parts. We will show that every solution of functional equation $Df(x,y) = 0$ is an additive-quadratic-quartic functional equation and we introduce a strictly contractive mapping which allows me to use the fixed point theory in the sense of L. Cădariu and V. Radu([2, 3, 4]) (See also [9, 10, 11, 12, 14, 15]). And then we can adopt the fixed point method for proving the stability of the functional equation $Df(x,y) = 0$.

Namely, starting from the given mapping f that approximately satisfies the functional equation $Df(x,y)=0$, a solution F of the functional equation $Df(x,y)=0$ is explicitly constructed by using the formula

$$F(x)=\lim_{n\to\infty}\left(\frac{f_o(3^n x)}{3^n}+\sum_{i=0}^{n} {}_nC_i\frac{(-1)^{n-i}(90)^i}{729^n}f_e(3^{2n-i}x)\right)$$

or

$$F(x)=\lim_{n\to\infty}3^n\left(f_o\left(\frac{x}{3^n}\right)+\sum_{i=0}^{n} {}_nC_i 90^i(-729)^{n-i}f_e\left(\frac{x}{3^{2n-i}}\right)\right),$$

which approximates the mapping f.

2. Main theorems

We recall the following result of the fixed point theory by Margolis and Diaz.

THEOREM 2.1. ([5] or [19]) *Suppose that a complete generalized metric space (X,d), which means that the metric d may assume infinite values, and a strictly contractive mapping $J: X\to X$ with the Lipschitz constant $0<L<1$ are given. Then, for each given element $x\in X$, either*

$$d(J^n x, J^{n+1}x)=+\infty,\ \forall n\in\mathbb{N}\cup\{0\},$$

or there exists a nonnegative integer k such that:
(1) $d(J^n x, J^{n+1}x)<+\infty$ for all $n\ge k$;
(2) the sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in $Y:=\{y\in X, d(J^k x,y)<+\infty\}$;
(4) $d(y,y^*)\le(1/(1-L))d(y,Jy)$ for all $y\in Y$.

The following theorem is a particular case of Baker's theorem [1] when $\delta=0$.

THEOREM 2.2. (Theorem 1 in [1]) *Suppose that V and W are vector spaces over $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ and $\alpha_0,\beta_0,\dots,\alpha_m,\beta_m$ are scalar such that $\alpha_j\beta_l-\alpha_l\beta_j\ne 0$ whenever $0\le j<l\le m$. If $f_l: V\to W$ for $0\le l\le m$ and*

$$\sum_{l=0}^{m} f_l(\alpha_l x+\beta_l y)=0$$

for all $x,y\in V$, then each f_l is a "generalized" polynomial mapping of "degree" at most $m-1$.

Baker [1] also states that if f is a "generalized" polynomial mapping of "degree" at most $m-1$, then f is expressed as $f(x) = x_0 + \sum_{l=1}^{m-1} a_l^*(x)$ for $x \in V$, where a_l^* is a monomial mapping of degree l and f has a property $f(rx) = x_0 + \sum_{l=1}^{m-1} r^l a_l^*(x)$ for $x \in V$ and $r \in \mathbb{Q}$. The monomial mapping of degree 1, 2 and 4 are also called an additive mapping, a quadric mapping and a quartic mapping, respectively.

THEOREM 2.3. *A mapping $f : V \to W$ satisfies $Df(x, y) = 0$ for all $x, y \in V$ with $f(0) = 0$ if and only if f is an additive-quadratic-quartic mapping.*

Proof. First, we assume that a mapping $f : V \to W$ satisfies $Df(x, y) = 0$ for all $x, y \in V$. Since the equalities $f_e(9x) - 90f_e(3x) + 729f_e(x) = 0$ and $f_o(3x) = 3f_o(x)$ are obtained from

$$\begin{aligned} f_e(9x) - 90f_e(3x) + 729f_e(x) =& Df_e(0, 3x) + 6Df_e(0, 2x) \\ &+ 36Df_e(x, x) + 75Df_e(0, x), \\ f_o(3x) - 3f_o(x) =& 2Df_o(-x, x) + Df_o(0, -x) \end{aligned}$$

for all $x \in V$, we can say that $Df_o(x, y) = 0$, $Dg(x, y) = 0$,$Dh(x, y) = 0$, $g(3x) = 3^4 g(x)$ and $h(3x) = 3^2(x)$ and $f_o(3x) = 3f_o(x)$, where g, h are defined by $g(x) := f_e(3x) - 3^2 f_e(x)$ and $h(x) := f_e(3x) - 3^4 f_e(x)$. Therefore, by the comments mentioned after Theorem 2.2, we conclude that f_o, g and h are an additive mapping and a quadratic mapping and a quartic mapping, respectively. With the equality $f(x) = f_o(x) + \frac{g(x)}{72} + \frac{-h(x)}{72}$, we obtain that f is an additive-quartic-quadratic mapping.

Conversely, assume that f_1, f_2, f_3 are mappings such that the equalities $f(x) := f_1(x) + f_2(x) + f_3(x)$, $Af_1(x, y) = 0$, $Qf_2(x, y) = 0$, and $Q'f_3(x, y) = 0$ hold for all $x, y \in V$. Then the equalities $f_1(x) = -f_1(-x)$, $f_2(x) = f_2(-x)$, $f_3(x) = f_3(-x)$, $f_1(2x) = 2f_1(x)$, $f_2(2x) = 4f_2(x)$, and $f_3(2x) = 16f_3(x)$ hold for all $x \in V$. From the above equalities, we obtain the equalities

$$\begin{aligned} Df_1(x, y) =& -Af_1(x + 3y, x + y) + 3Af_1(x + 2y, x) \\ &- 3Af_1(x + y, x - y) + Af_1(x, x - 2y), \\ Df_2(x, y) =& -\frac{Qf_2(x + 3y, x + y)}{2} + \frac{3Qf_2(x + 2y, x)}{2} \\ &- \frac{3Qf_2(x + y, x - y)}{2} + \frac{Qf_2(x, x - 2y)}{2}, \\ Df_3(x, y) =& Q'f_3(x + y, y) - Q'f_3(x, y) \end{aligned}$$

for all $x, y \in V$, which mean that

$$Df(x,y) = Df_1(x,y) + Df_2(x,y) + Df_3(x,y) = 0$$

as we desired. □

In the following theorem, we can prove the generalized Hyers-Ulam stability of the functional equation $Df(x,y) = 0$ for all $x, y \in V$ by using the fixed point method.

THEOREM 2.4. *Let $f : V \to Y$ be a mapping for which there exists a mapping $\varphi : V^2 \to [0,\infty)$ such that the inequality*

$$\|Df(x,y)\| \le \varphi(x,y) \tag{2.1}$$

holds for all $x, y \in V$. If there exists a constant $0 < L < 1$ such that φ has the property

$$\varphi(3x,3y) \le (\sqrt{59778} - 243)L\varphi(x,y) \tag{2.2}$$

for all $x, y \in V$, then there exists a unique solution mapping $F : V \to Y$ of $DF(x,y) = 0$ such that

$$\|f(x) - f(0) - F(x)\| \le \frac{\Phi(x)}{729(1-L)} \tag{2.3}$$

for all $x \in V$ with $F(0) = 0$, where $\Phi(x) = \varphi_e(0,3x) + 6\varphi_e(0,2x) + 36\varphi_e(x,x) + 75\varphi_e(0,x) + 486\varphi_e(x,-x) + 243\varphi_e(0,x)$. In particular, F is represented by

$$F(x) = \lim_{n\to\infty}\left(\frac{f_o(3^n x)}{3^n} + \sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}(90)^i}{729^n}(f_e(3^{2n-i}x) - f(0))\right) \tag{2.4}$$

for all $x \in V$.

Proof. Let $\tilde{f} : V \to Y$ be the mapping defined by $\tilde{f}(x) := f(x) - f(0)$. Then $D\tilde{f}(x,y) = Df(x,y)$ for all $x, y \in V$ and $\tilde{f}(0) = 0$. Let S be the set of all mappings $g : V \to Y$ with $g(0) = 0$. We introduce a generalized metric on S by

$$d(g,h) = \inf\left\{K \in \mathbb{R}_+ \middle| \, \|g(x) - h(x)\| \le K\Phi(x) \text{ for all } x \in V\right\}.$$

It is easy to show that (S,d) is a generalized complete metric space. Now we consider the mapping $J : S \to S$, which is defined by

$$Jg(x) := -\frac{g(9x)}{1458} - \frac{g(-9x)}{1458} + \frac{333g(3x)}{1458} - \frac{153g(-3x)}{1458}$$

for all $x \in V$. Notice that the equality

$$J^n g(x) = \frac{g_o(3^n x)}{3^n} + \sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}(90)^i}{729^n} g_e(3^{2n-i}x)$$

holds for all $n \in \mathbb{N}$ and $x \in V$. Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \le K$. From the definition of d, we have

$$\begin{aligned}
\|Jg(x) - Jh(x)\| \le& \frac{1}{1458}\|g(9x) - h(9x)\| + \frac{1}{1458}\|g(-9x) - h(-9x)\| \\
&+ \frac{333}{1458}\|g(3x) - h(3x)\| + \frac{153}{1458}\|g(-3x) - h(-3x)\| \\
\le& \frac{\Phi(9x)K}{729} + \frac{\Phi(3x)K}{3} \\
\le& \frac{(\sqrt{59778} - 243)KL\Phi(3x)}{729} + \frac{K\Phi(3x)}{3} \\
\le& \frac{(\sqrt{59778} - 243)^2}{729}KL^2\Phi(x) + \frac{\sqrt{59778} - 243}{3}KL\Phi(x)) \\
\le& \frac{(\sqrt{59778} - 243)^2 + 486(\sqrt{59778} - 243)}{729}KL\Phi(x) \\
=& KL\Phi(x)
\end{aligned}$$

for all $x \in V$, which implies that

$$d(Jg, Jh) \le Ld(g, h)$$

for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L. Moreover, by (2.1) we see that

$$\begin{aligned}
\|\tilde{f}(x) - J\tilde{f}(x)\| =& \frac{1}{729}\|Df_e(0, 3x) + 6Df_e(0, 2x) + 36Df_e(x, x) \\
&+ 75Df_e(0, x) + 486Df_o(x, -x) + 243Df_o(0, x)\| \\
\le& \frac{1}{729}(\varphi_e(0, 3x) + 6\varphi_e(0, 2x) + 36\varphi_e(x, x) + 75\varphi_e(0, x) \\
&+ 486\varphi_e(x, -x) + 243\varphi_e(0, x)) \\
\le& \frac{\Phi(x)}{729}
\end{aligned}$$

for all $x \in V$. It means that $d(\tilde{f}, J\tilde{f}) \le \frac{1}{729} < \infty$ by the definition of d. Therefore according to Theorem 2.1, the sequence $\{J^n\tilde{f}\}$ converges to the unique fixed point $F : V \to Y$ of J in the set $T = \{g \in S | d(\tilde{f}, g) <$

$\infty\}$, which is represented by (2.4) for all $x \in V$. Notice that

$$d(\tilde{f}, F) \le \frac{1}{1-L} d(\tilde{f}, J\tilde{f}) \le \frac{1}{729(1-L)},$$

which implies (2.3). By the definition of F, together with (2.1) and (2.2), we have

$$\begin{aligned}
&\|DF(x,y)\| \\
&= \lim_{n\to\infty} \|DJ^n \tilde{f}(x,y)\| \\
&= \lim_{n\to\infty} \left\| \frac{Df_o(3^n x, 3^n y)}{3^n} + \sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i} 90^i}{729^n} Df_e(3^{2n-i}x, 3^{2n-i}y) \right\| \\
&\le \lim_{n\to\infty} \left(\frac{\varphi_e(3^n x, 3^n y)}{3^n} + \frac{1}{729^n} \sum_{i=0}^{n} {}_nC_i 90^i \varphi_e(3^{2n-i}x, 3^{2n-i}y) \right) \\
&\le \lim_{n\to\infty} \left(\frac{1}{3^n} + \frac{1}{729^n} \sum_{i=0}^{n} {}_nC_i (\sqrt{59778} - 243)^{n-i} L^{n-i} 90^i \right) \varphi_e(3^n x, 3^n y) \\
&\le \lim_{n\to\infty} \left(\frac{1}{3^n} + \frac{1}{729^n} \left((\sqrt{59778} - 243)L + 90 \right)^n \right) \varphi_e(3^n x, 3^n y) \\
&\le \lim_{n\to\infty} \left(\left(\frac{243}{729} \right)^n + \frac{1}{729^n} \left(\sqrt{59778} - 243 + 90 \right)^n \right) \varphi_e(3^n x, 3^n y) \\
&\le \lim_{n\to\infty} \left(\left(\frac{\sqrt{59778} + 243}{729} \right)^n + \left(\frac{\sqrt{59778} + 243}{729} \right)^n \right) \varphi_e(3^n x, 3^n y) \\
&\le 2 \lim_{n\to\infty} \left(\frac{(\sqrt{59778} + 243)(\sqrt{59778} - 243)}{729} \right)^n L^n \varphi_e(x,y) \\
&= 2 \lim_{n\to\infty} L^n \varphi_e(x,y) \\
&= 0
\end{aligned}$$

for all $x, y \in V$ i.e., F is a solution of the functional equation $DF(x,y) = 0$ and $F(0) = 0$. Notice that if F is a solution of the functional equation $DF(x,y) = 0$ with $F(0) = 0$, then the equality

$$\begin{aligned}
F(x) - JF(x) = \frac{1}{729} (&DF_e(0,3x) + 6DF_e(0,2x) + 36DF_e(x,x) \\
&+ 75DF_e(0,x) + 486DF_o(x,-x) + 243DF_o(0,x))
\end{aligned}$$

implies that F is a fixed point of J. □

THEOREM 2.5. *Let $f : V \to Y$ be a mapping for which there exists a mapping $\varphi : V^2 \to [0,\infty)$ such that the inequality (2.1) holds for all*

$x, y \in V$. If there exists a constant $0 < L < 1$ such that φ has the property

$$L\varphi(3x, 3y) \geq \frac{729}{\sqrt{2754} - 45}\varphi(x, y) \tag{2.5}$$

for all $x, y \in V$, then there exists a unique solution mapping $F : V \to Y$ of $DF(x, y) = 0$ such that

$$\|f(x) - f(0) - F(x)\| \leq \frac{\Psi(x)}{1 - L} \tag{2.6}$$

for all $x \in V$ with $F(0) = 0$, where $\Psi(x)$ is given by

$$\begin{aligned}\Psi(x) :=&\varphi_e\left(0, \frac{x}{3}\right) + 6\varphi_e\left(0, \frac{2x}{9}\right) + 36\varphi_e\left(\frac{x}{9}, \frac{x}{9}\right) + 75\varphi_e\left(0, \frac{x}{9}\right)\\ &+ 2\varphi_e\left(\frac{-x}{3}, \frac{x}{3}\right) + \varphi_e\left(0, \frac{-x}{3}\right).\end{aligned}$$

In particular, F is represented by

$$F(x) = \lim_{n\to\infty}\left(3^n f_o\left(\frac{x}{3^n}\right) + \sum_{i=0}^{n} {}_nC_i 90^i(-729)^{n-i}\left(f_e\left(\frac{x}{3^{2n-i}}\right)\right) - f(0)\right) \tag{2.7}$$

for all $x \in V$.

Proof. Let the mapping $\tilde{f}$ and the set S be as in the proof of Theorem 2.3 with a generalized metric d on S given by

$$d(g, h) = \inf\left\{K \in \mathbb{R}_+ \middle| \|g(x) - h(x)\| \leq K\Psi(x) \text{ for all } x \in V\right\}.$$

Now we consider the mapping $J : S \to S$ defined by

$$Jg(x) := \frac{1}{2}\left(93g\left(\frac{x}{3}\right) + 87g\left(\frac{-x}{3}\right) - 729g\left(\frac{x}{9}\right) - 729g\left(\frac{-x}{9}\right)\right)$$

for all $x \in V$. Notice that the equality

$$J^n g(x) = 3^n g_o\left(\frac{x}{3^n}\right) + \sum_{i=0}^{n} {}_nC_i 90^i(-729)^{n-i} g_e\left(\frac{x}{3^{2n-i}}\right)$$

holds for all $n \in \mathbb{N}$ and $x \in V$. Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g,h) \le K$. From the definition of d, we have

$$\begin{aligned}\|Jg(x) - Jh(x)\| \le &\frac{1}{2}\left(93\left\|g\left(\frac{x}{3}\right) - h\left(\frac{x}{3}\right)\right\| + 87\left\|g\left(\frac{-x}{3}\right) - h\left(\frac{-x}{3}\right)\right\|\right. \\ &\left. + 729\left\|g\left(\frac{x}{9}\right) - h\left(\frac{x}{9}\right)\right\| + 729\left\|g\left(\frac{-x}{9}\right) - h\left(\frac{-x}{9}\right)\right\|\right) \\ \le &729K\Psi\left(\frac{x}{9}\right) + 90K\Psi\left(\frac{x}{3}\right) \\ \le &L^2\frac{(\sqrt{2754}-45)^2}{729}K\Psi(x) + \frac{90(\sqrt{2754}-45)}{729}LK\Psi(x) \\ \le &LK\Psi(x)\end{aligned}$$

for all $x \in V$, which implies that

$$d(Jg, Jh) \le Ld(g,h)$$

for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L. Moreover, by (2.1) we see that

$$\begin{aligned}\|\tilde{f}(x) - J\tilde{f}(x)\| = &\left\|Df_e\left(0, \frac{x}{3}\right) + 6Df_e\left(0, \frac{2x}{9}\right) + 36Df_e\left(\frac{x}{9}, \frac{x}{9}\right)\right. \\ &\left. + 75Df_e\left(0, \frac{x}{9}\right) + 2Df_o\left(\frac{-x}{3}, \frac{x}{3}\right) + Df_o\left(0, \frac{-x}{3}\right)\right\| \\ \le &\varphi_e\left(0, \frac{x}{3}\right) + 6\varphi_e\left(0, \frac{2x}{9}\right) + 36\varphi_e\left(\frac{x}{9}, \frac{x}{9}\right) + 75\varphi_e\left(0, \frac{x}{9}\right) \\ &+ 2\varphi_e\left(\frac{-x}{3}, \frac{x}{3}\right) + \varphi_e\left(0, \frac{-x}{3}\right) \\ = &\Psi(x)\end{aligned}$$

for all $x \in V$. It means that $d(\tilde{f}, J\tilde{f}) \le 1 < \infty$ by the definition of d. Therefore according to Theorem 2.1, the sequence $\{J^n\tilde{f}\}$ converges to the unique fixed point $F : V \to Y$ of J in the set $T = \{g \in S | d(\tilde{f}, g) < \infty\}$, which is represented by (2.7) for all $x \in V$. Notice that

$$d(\tilde{f}, F) \le \frac{1}{1-L}d(\tilde{f}, J\tilde{f}) \le \frac{1}{1-L},$$

which implies (2.6). By the definition of F, together with (2.1) and (2.5), we have

$$
\begin{aligned}
&\|DF(x,y)\|\\
&= \lim_{n\to\infty} \|DJ^n f(x,y)\|\\
&= \lim_{n\to\infty} \left\| 3^n f_o\left(\frac{x}{3^n}, \frac{y}{3^n}\right) + \sum_{i=0}^{n} {}_nC_i 90^{n-i}(-729)^i f_e\left(\frac{x}{3^{2n-i}}, \frac{y}{3^{2n-i}}\right)\right\|\\
&\le \lim_{n\to\infty} \left(3^n \varphi_e\left(\frac{x}{3^n}, \frac{y}{3^n}\right) + \sum_{i=0}^{n} {}_nC_i 729^{n-i}90^i \varphi_e\left(\frac{x}{3^{2n-i}}, \frac{y}{3^{2n-i}}\right)\right)\\
&\le \lim_{n\to\infty} \big(3^n + \sum_{i=0}^{n} {}_nC_i 90^i(\sqrt{2754}-45)^{n-i}L^{n-i}\big)\varphi_e\left(\frac{x}{3^n}, \frac{y}{3^n}\right)\\
&\le \lim_{n\to\infty} \big(3^n + ((\sqrt{2754}-45)+90)^n\big)\varphi_e\left(\frac{x}{3^n}, \frac{y}{3^n}\right)\\
&\le \lim_{n\to\infty} \big(((\sqrt{2754}+45))^n + (\sqrt{2754}+45))^n\big)\varphi_e\left(\frac{x}{3^n}, \frac{y}{3^n}\right)\\
&\le 2\lim_{n\to\infty} L^n \frac{((\sqrt{2754}+45))^n(\sqrt{2754}-45))^n}{729^n}\varphi_e(x,y)\\
&\le 2\lim_{n\to\infty} L^n \varphi_e(x,y)\\
&=0
\end{aligned}
$$

for all $x, y \in V$ i.e., F is a solution of the functional equation $DF(x,y) = 0$ with $F(0) = 0$. Notice that if F is a solution of the functional equation $DF(x,y) = 0$ with $F(0) = 0$, then the equality $F(x) - JF(x) = DF_e\big(0, \frac{x}{3}\big) + 6DF_e\big(0, \frac{2x}{9}\big) + 36DF_e\big(\frac{x}{9}, \frac{x}{9}\big) + 75DF_e\big(0, \frac{x}{9}\big) + 2DF_o\big(\frac{-x}{3}, \frac{x}{3}\big) + DF_o\big(0, \frac{-x}{3}\big)$ implies that F is a fixed point of J. □

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