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# CHARACTERIZATIONS OF THE EXPONENTIAL DISTRIBUTION BY RECORD VALUES

Se-Kyung Chang* and Min-Young Lee**

Abstract. This paper presents characterizations based on the identical distribution and the finite moments of the exponential distribution by record values. We prove that $X \in EXP(\sigma)$, $\sigma > 0$, if and only if $X_{U(n+k)} - X_{U(n)}$ and $X_{U(n)} - X_{U(n-k)}$ for $n > 1$ and $k \geq 1$ are identically distributed. Also, we show that $X \in EXP(\sigma)$, $\sigma > 0$, if and only if $E\left(X_{U(n+k)} - X_{U(n)}\right) = E\left(X_{U(n)} - X_{U(n-k)}\right)$ for $n > 1$ and $k \geq 1$.

## 1. Introduction

Let $\{X_n,\ n \geq 1\}$ be a sequence of independent and identically distributed(i.i.d.) random variables with cumulative distribution function(cdf) $F(x)$ and probability density function(pdf) $f(x)$. Suppose

$$Y_n = \max\{X_1, X_2, \cdots, X_n\}$$

for $n \geq 1$. We say $X_j$ is an upper record value of this sequence, if $Y_j > Y_{j-1}$ for $j > 1$. By definition, $X_1$ is an upper as well as a lower record value.

The indices at which the upper record values occur are given by the record times $\{U(n),\ n \geq 1\}$, where $U(n) = \min\{j | j > U(n-1)\,, X_j > X_{U(n-1)}\,, n \geq 2\}$ with $U(1) = 1$.

We assume that all upper record values $X_{U(i)}$ for $i \geq 1$ occur at a sequence $\{X_n,\ n \geq 1\}$ of i.i.d. random variables.

A continuous random variable $X$ has the exponential distribution with parameter $\sigma > 0$ if it has a cdf $F(x)$ of the form

$$F(x) = \begin{cases} 1 - e^{-\frac{x}{\sigma}},\ x > 0,\ \sigma > 0 \\ \\ 0, \text{ otherwise.} \end{cases} \tag{1}$$

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A notation that designates that $X$ has the cdf (1) is $X \in EXP(\sigma)$.

Some characterizations based on the identical distribution are known. Ahsanullah [1, 2] characterized that $X \in EXP(\sigma)$, $\sigma > 0$ if and only if $X_{U(n)}$ and $Z_1 + Z_2 + \cdots + Z_n$ for $n > 1$ are identically distributed, where $Z_1, Z_2, \cdots, Z_n$ are i.i.d. $EXP(\sigma)$. And Ahsanullah [1, 2] proved that $X \in EXP(\sigma)$, $\sigma > 0$, if and only if $X_{U(n)} - X_{U(m)}$ and $X_{U(n-m)}$ for $1 < m < n$ are identically distributed. Moreover Ahsanullah [1, 2] showed that $X \in EXP(\sigma)$, $\sigma > 0$, if and only if $X_{U(n+1)} - X_{U(n)}$ and $X_{U(n)} - X_{U(n-1)}$ for $n > 1$ are identically distributed and $X \in EXP(\sigma)$, $\sigma > 0$, if and only if $E\left[X_{U(n+1)} - X_{U(n)}\right] = E\left[X_{U(n)} - X_{U(n-1)}\right]$ for $n > 1$. We extend the above results from the record times $(U(n-1), U(n), U(n+1))$ to the record times $(U(n-k), U(n), U(n+k))$.

In this paper, we will give characterizations based on the identical distribution and the expectation of the exponential distribution by record values.

## 2. Main results

Theorem 1. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf $F(x)$ which is an absolutely continuous with pdf $f(x)$ and $F(x) < 1$ for $x > 0$. Then $F(x) = 1 - e^{-\frac{x}{\sigma}}$ for $x > 0$ and $\sigma > 0$, if and only if $X_{U(n+k)} - X_{U(n)}$ and $X_{U(n)} - X_{U(n-k)}$ for $n > 1$ and $k \geq 1$ are identically distributed and $F(x)$ belongs to $C_2$.*

*Proof.* If $F(x) = 1 - e^{-\frac{x}{\sigma}}$ for $x > 0$ and $\sigma > 0$, then it can easily be seen that $X_{U(n+k)} - X_{U(n)}$ and $X_{U(n)} - X_{U(n-k)}$ for $n > 1$ and $k > 1$ are identically distributed.

We will prove the sufficient condition. The joint pdf $f_{n,\,n+k}(x, y)$ of $X_{U(n)}$ and $X_{U(n+k)}$ is

$$f_{n,\,n+k}(x, y) = \frac{1}{\Gamma(n)\Gamma(k)} \left(R(x)\right)^{n-1} r(x) \left(R(y) - R(x)\right)^{k-1} f(y)$$

for $0 < x < y$, $n > 1$ and $k \geq 1$.

Consider the functions $V = X_{U(n+k)} - X_{U(n)}$ and $W = X_{U(n)}$. It follows that $x_{U(n)} = w$, $x_{U(n+k)} = v + w$ and $|J| = 1$. Thus we can write the joint pdf $f_{V,W}(v, w)$ of $V$ and $W$ as

$$f_{V,W}(v,w) = \frac{1}{\Gamma(n)\Gamma(k)} \left(R(w)\right)^{n-1} r(w) \left(R(v+w) - R(w)\right)^{k-1} f(v+w)$$

for $v$, $w > 0$, $n > 1$ and $k \geq 1$.

The marginal pdf $f_V(v)$ of $V$ is given by

$$\begin{aligned} f_V(v) &= \int_0^\infty f_{V,W}(v,w)\, dw \\ &= \int_0^\infty \frac{1}{\Gamma(n)\Gamma(k)} \left(R(w)\right)^{n-1} r(w) \left(R(v+w) - R(w)\right)^{k-1} f(v+w) dw \end{aligned} \tag{2}$$

for $v > 0$, $n > 1$ and $k \geq 1$.

Also, the joint pdf $f_{n-k,\,n}(x, y)$ of $X_{U(n-k)}$ and $X_{U(n)}$ is

$$f_{n-k,\,n}(x,y) = \frac{1}{\Gamma(n-k)\Gamma(k)} \left(R(x)\right)^{n-k-1} r(x) \left(R(y) - R(x)\right)^{k-1} f(y)$$

for $0 < x < y$, $n > 1$ and $k \geq 1$.

Let us use the transformation $V = X_{U(n)} - X_{U(n-k)}$ and $W = X_{U(n-k)}$. The Jacobian of the transformation is $|J| = 1$. Thus we can write the joint pdf $f_{V,W}(v, w)$ of $V$ and $W$ as

$$f_{V,W}(v,w) = \frac{1}{\Gamma(n-k)\Gamma(k)} \left(R(w)\right)^{n-k-1} r(w) \left(R(v+w) - R(w)\right)^{k-1} f(v+w)$$

for $v$, $w > 0$, $n > 1$ and $k \geq 1$.

The marginal pdf $f_V(v)$ of $V$ is given by

$$\begin{aligned} f_V(v) &= \int_0^\infty f_{V,W}(v,w)\, dw \\ &= \int_0^\infty \frac{1}{\Gamma(n-k)\Gamma(k)} \left(R(w)\right)^{n-k-1} r(w) \\ &\qquad \times \left(R(v+w) - R(w)\right)^{k-1} f(v+w) dw \end{aligned} \tag{3}$$

for $v > 0$, $n > 1$ and $k \geq 1$.

Since $X_{U(n+k)} - X_{U(n)}$ and $X_{U(n)} - X_{U(n-k)}$ for $n > 1$ and $k \geq 1$ are identically distributed, by using (2) and (3), we get

$$\begin{aligned}\int_0^\infty \frac{1}{\Gamma(n)\Gamma(k)} \left(R(w)\right)^{n-1} r(w) \left(R(v+w) - R(w)\right)^{k-1} f(v+w)dw \\ = \int_0^\infty \frac{1}{\Gamma(n-k)\Gamma(k)} \left(R(w)\right)^{n-k-1} r(w) \\ \times \left(R(v+w) - R(w)\right)^{k-1} f(v+w)dw\end{aligned} \tag{4}$$

for $v > 0$, $n > 1$ and $k \geq 1$.

From (4), we get on simplification

$$\begin{aligned}\int_0^\infty \left(R(w)\right)^{n-k-1} r(w) \left(R(v+w) - R(w)\right)^{k-1} f(v+w) \\ \times \left(\frac{(R(w))^k}{\Gamma(n)} - \frac{1}{\Gamma(n-k)}\right) dw = 0\end{aligned} \tag{5}$$

for $v > 0$, $n > 1$ and $k \geq 1$.

We know that $r(x)$ and $R(x)$ are greater than zero. In addition, we say $F(x)$ belongs to the class $C_2$ if the hazard rate $r(x)$ is either monotone increasing or decreasing. Consequently $R(v+w) \leq R(w)$ or $R(v+w) \geq R(w)$. Therefore $R(v+w) - R(w) \leq 0$ or $R(v+w) - R(w) \geq 0$.

Thus if $F(x) \in C_2$, then (5) is true if

$$R(v+w) - R(w) = 0$$

for $w$ and any fixed $v > 0$.

Hence

$$r(v+w) = r(w)$$

for $w$ and any fixed $v > 0$.

By the characterization property of the exponential distribution, we have

$$F(x) = 1 - e^{-\frac{x}{\sigma}}$$

for $x > 0$ and $\sigma > 0$.

This completes the proof. □

THEOREM 2. *Let* $\{X_n, n \geq 1\}$ *be a sequence of i.i.d. random variables with cdf* $F(x)$ *which is an absolutely continuous with pdf* $f(x)$ *and* $F(x) < 1$ *for* $x > 0$*. Then* $F(x) = 1 - e^{-\frac{x}{\sigma}}$ *for* $x > 0$ *and* $\sigma > 0$*, if and only if* $E\left(X_{U(n+k)} - X_{U(n)}\right) = E\left(X_{U(n)} - X_{U(n-k)}\right)$ *for* $n > 1$ *and* $k \geq 1$ *is finite and* $F(x)$ *belongs to* $C_2$*.*

*Proof.* If $F(x) = 1 - e^{-\frac{x}{\sigma}}$ for $x > 0$ and $\sigma > 0$, then it can easily be seen that $E\left(X_{U(n+k)} - X_{U(n)}\right) = E\left(X_{U(n)} - X_{U(n-k)}\right)$ for $n > 1$ and $k > 1$.

We will prove the sufficient condition. From (2) of Theorem 1, we can write the expectation of $V = X_{U(n+k)} - X_{U(n)}$ as

$$\begin{aligned} E\left(X_{U(n)} - X_{U(n-k)} = V\right) &= \int_0^\infty v\, f_V(v)\ dv \\ &= \int_0^\infty v\Big(\int_0^\infty \frac{1}{\Gamma(n)\Gamma(k)}(R(w))^{n-1} r(w) \\ &\qquad \times (R(v+w) - R(w))^{k-1} f(v+w)dw\Big)dv \\ &= \int_0^\infty \int_0^\infty \frac{1}{\Gamma(n)\Gamma(k)}\left(R(w)\right)^{n-1} r(w) \\ &\qquad \times \left(R(v+w) - R(w)\right)^{k-1} f(v+w) v\, dwdv \end{aligned} \tag{6}$$

for $n > 1$ and $k \geq 1$.

In the same manner as (6), we can write the expectation of $V = X_{U(n)} - X_{U(n-k)}$ as

$$\begin{aligned} &E(X_{U(n)} - X_{U(n-k)} = V) \\ &= \int_0^\infty v\, f_V(v)\ dv \\ (7) &= \int_0^\infty u \left( \int_0^\infty \frac{\left(R(w)\right)^{n-k-1} r(w) \left(R(v+w) - R(w)\right)^{k-1} f(v+w)}{\Gamma(n-k)\Gamma(k)} dw \right) dv \\ &= \int_0^\infty \int_0^\infty \frac{\left(R(w)\right)^{n-k-1} r(w) \left(R(v+w) - R(w)\right)^{k-1} f(v+w) v}{\Gamma(n-k)\Gamma(k)}\, dwdv \end{aligned}$$

for $n > 1$ and $k \geq 1$.

Since $E\left(X_{U(n+k)} - X_{U(n)}\right) = E\left(X_{U(n)} - X_{U(n-k)}\right)$ for $n > 1$ and $k \geq 1$

is finite, by using (6) and (7), we get

$$\begin{aligned}(8)\quad &\int_0^\infty \int_0^\infty \frac{1}{\Gamma(n)\Gamma(k)} \left(R(w)\right)^{n-1} r(w) \\ &\qquad \times \left(R(v+w) - R(w)\right)^{k-1} f(v+w)v \, dwdv \\ &= \int_0^\infty \int_0^\infty \frac{\left(R(w)\right)^{n-k-1} r(w) \left(R(v+w)-R(w)\right)^{k-1} f(v+w)v}{\Gamma(n-k)\Gamma(k)} \, dwdv\end{aligned}$$

for $n > 1$ and $k \geq 1$.

From (8), we get on simplification

$$\begin{aligned}(9)\quad &\int_0^\infty \int_0^\infty \left(R(w)\right)^{n-k-1} r(w) \left(R(v+w) - R(w)\right)^{k-1} f(v+w)v \\ &\qquad \times \left( \frac{(R(w))^k}{\Gamma(n)} - \frac{1}{\Gamma(n-k)} \right) dwdv = 0\end{aligned}$$

for $n > 1$ and $k \geq 1$.

In the same manner as Theorem 1, if $F(x) \in C_2$, then (9) is true if

$$R(v+w) - R(w) = 0$$

for $w$ and any fixed $v > 0$.

Hence

$$r(v+w) = r(w)$$

for $w$ and any fixed $v > 0$.

By the characterization property of the exponential distribution, we have

$$F(x) = 1 - e^{-\frac{x}{\sigma}}$$

for $x > 0$ and $\sigma > 0$.

This completes the proof. □

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Department of Mathematics Education
Cheongju University
Cheongju, Chungbuk 360-764, Republic of Korea
*E-mail*: skchang@cju.ac.kr

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Department of Applied Mathematics
Dankook University
Cheonan, Chungnam 330-714, Republic of Korea
*E-mail*: leemy@dku.edu