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INCOMPRESSIBLE NAVIER-STOKES EQUATIONS IN HETEROGENEOUS MEDIA

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Abstract. The homogenization of non-stationary Navier-Stokes equations on anisotropic heterogeneous media is investigated. The effective coefficients of the homogenized equations are found. It is pointed out that the resulting homogenized limit systems are of the same form of non-stationary Navier-Stokes equations with suitable coefficients. Also, steady Stokes equations as cell problems are identified.

A compactness theorem is proved in order to deal with time dependent homogenization problems.

1. Introduction

In this paper, we address the problem of the homogenization of non-stationary Navier-Stokes equations;

$$\rho \frac{\partial}{\partial t}\mathbf{u} + (\mathbf{u}, \nabla)\mathbf{u} + \mathcal{E}(\mathbf{u}) + \nabla p = \mathbf{f}, \qquad \operatorname{div} \mathbf{u} = 0$$

in heterogeneous media. Here $\mathbf{u}(x,t) = (u^1, u^2, \cdots, u^n)$ is the Eulerian velocity of a fluid flow and $(\mathbf{u}, \nabla)u_k = \sum_{i=1}^n u_i \frac{\partial}{\partial x_i} u_k$, $k = 1, 2, \cdots n$ (the *elasticity* operator $\mathcal{E}$ is described in Section 2).

The homogenized problems for fluid flows governed by steady Stokes equations and Navier-Stokes equations have been extensively studied by many authors - for example, C. Conca, T. Levy, Sanchez-Palencia, G. Allaire, A. Mikeli, Firdaouss-Guermond et al.

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Compared with other studies, we focus on the homogenization of *non-stationary* Navier-Stokes systems with distinct coefficients on *anisotropic* materials - these general situations provide a bunch of cell problems of matrix type. We identify that the limit systems are of the same form of non-stationary Navier-Stokes equations with *effective coefficients*. It is also very interesting to get *accessible* cell problems - we find the *classical Stokes equations* on a flat torus as the cell problems which provide the homogenized effective elasticity tensor.

In Section 2, we construct the ε-model and present terminology to explain well-posedness and a-priori estimates. In Section 3, we briefly review the two-scale convergence, and introduce the two-scale limit of the stain operator, and prove a compactness theorem which is useful to deal with time dependent problems. In the last section, the homogenized equations for the ε-model of non-stationary Navier-Stokes equations are investigated .

2. The ε-model

We consider a structure consisting of fissures and matrices periodically distributed in a domain Ω in $\mathbf{R}^n$ with period εY, where $\varepsilon > 0$ and $Y \equiv [0,1]^n$ is the unit cube. Let Y be given in complementary parts, Y_1 and Y_2, which represent the fissure and matrix, respectively. Denote by $\chi_m(y)$ the characteristic function of Y_m for $m = 1, 2$, extended Y-periodically to all of $\mathbf{R}^n$. Thus, $\chi_1(y) + \chi_2(y) = 1$. We shall assume that the set $\{y \in \mathbf{R}^n : \chi_1(y) = 1\}$ is connected and smooth.

The domain Ω is thus divided into the two subdomains, Ω_1^ε and Ω_2^ε, representing the *fissure* and *matrix*, respectively, and given by

$$\Omega_m^\varepsilon = \left\{ x \in \Omega : \chi_m\left(\frac{x}{\varepsilon}\right) = 1 \right\}, \quad m = 1, 2.$$

Let $\Gamma_{12}^\varepsilon \equiv \partial\Omega_1^\varepsilon \cap \Omega_2^\varepsilon \cap \Omega$ be the part of the interface of Ω_1^ε with Ω_2^ε that is interior to Ω, and let $\Gamma_{12} \equiv \partial Y_1 \cap \partial Y_2 \cap Y$ be the corresponding part in the cell Y. Likewise, let $\Gamma_{22} \equiv Y_2 \cap \partial Y$ and denote by Γ_{22}^ε its periodic extension which forms the interface with those parts of the matrix Ω_2^ε which lie within neighboring εY- cells. These are the *local blocks* and we denote them by Y_2^ε.

We construct a system consisting of a non-stationary incompressible Navier-Stokes system in Ω_1^ε coupled across the interface Γ_{12}^ε to another non-stationary Navier-Stokes system in Ω_2^ε. The structure of fissured medium produces very high frequency spatial variations of pressures in the matrix and fissures, and so leads to corresponding variations of velocity fields.

In order to describe these, the fluid *pressures* in Ω_m^ε are denoted by $p_m^\varepsilon(x,t)$ at the position $x \in \Omega_m^\varepsilon$, $m = 1,2$ at time t. We let $\mathbf{u}_m = (u_{m1}, u_{m2}, \cdots, u_{mn})$ be the *velocity field* at $x \in \Omega_m^\varepsilon$, $m = 1,2$ and time t, and $\varepsilon_{kl}(\mathbf{u}) \equiv \frac{1}{2}\left(\frac{\partial}{\partial x_k}u_l + \frac{\partial}{\partial x_l}u_k\right)$ be the (linearized) *strain* tensor, which measures the *local deformation of velocity*. The *stress* $\sigma(\mathbf{u})$ is a necessarily symmetric tensor that represents the internal forces on surface elements resulting from such deformations, and we assume that the material is governed by the *generalized Hooke's law*

$$\sigma_{ij}^m(\mathbf{u}_m) = \sum_{k,l=1}^n a_{ijkl}^m \varepsilon_{kl}(\mathbf{u}_m), \quad m = 1,2.$$

The positive definite symmetric *elasticity* tensor a_{ijkl} provides a model for general anisotropic materials. We assume that $a_{ijkl}^m(x,y)$ $(m = 1,2)$ are bounded continuous functions such that

$$c_1 \sum_{i,j,k,l=1}^n \eta_{ij}\,\eta_{kl} \le \sum_{i,j,k,l=1}^n a_{ijkl}^m(x,y)\eta_{ij}\,\eta_{kl} \le c_2 \sum_{i,j,k,l=1}^n \eta_{ij}\,\eta_{kl}, \tag{1}$$

where (η_{ij}) is an arbitrary symmetric matrix, $x \in \Omega$ and $c_1, c_2 > 0$. The boundary conditions will involve with the surface density of forces or *traction* $\sum_{j=1}^n \sigma_{ij}\,n_j$ determined by the unit normal vector $\mathbf{n} = (n_1, n_2, \ldots, n_n)$ on any boundary or interface. The normal will be directed *out* of Ω_2^ε. The elastic structure is described by bilinear forms

$$e_m^\varepsilon(\mathbf{u},\mathbf{v}) \equiv \sum_{i,j,k,l=1}^n \int_{\Omega_m^\varepsilon} a_{ijkl}^m(x, \frac{x}{\varepsilon})\varepsilon_{kl}(\mathbf{u})\varepsilon_{ij}(\mathbf{v})dx,$$

$m = 1,2$, on the space

$$\mathbf{V} \equiv \left\{\mathbf{v} \in H^1(\Omega)^n : \mathbf{v} = 0 \text{ on } \Gamma_0\right\}, \qquad \Gamma_0 \subset \partial\Omega$$

of *admissible velocity fields of fluid*. The local description is obtained by means of Green's theorem

$$e_m^\varepsilon(\mathbf{u},\mathbf{v}) = \int_{\Omega_m^\varepsilon} \mathcal{E}_m^\varepsilon(\mathbf{u}(x))\cdot\mathbf{v}(x)dx + (-1)^m \sum_{i,j=1}^n \int_{\Gamma_{12}^\varepsilon} \sigma_{ij}^m\Big(x, \frac{x}{\varepsilon}\Big)(\mathbf{u}(s))n_j v_i(s)dS,$$

where the formal operator is given by

$$\mathcal{E}_m^\varepsilon(\mathbf{u})_i = -\sum_{j=1}^n \sum_{k,l=1}^n \partial_j \, a_{ijkl}^m(x, \frac{x}{\varepsilon}) \, \varepsilon_{kl}(\mathbf{u}), \quad 1 \le i \le n, \quad m = 1, 2,$$

whenever $\mathbf{u}, \mathbf{v} \in \mathbf{V}$ and $\mathcal{E}_m(\mathbf{u}) \in [L^2(\Omega_m^\varepsilon)]^n$.

Now, we are in the position to present the ε-model originated from the high frequency spatial variations of pressures and velocity fields in the matrix and fissures. The ε-model for diffusion on a *Navier-Stokes fissured medium* is as follows:

$$\text{(1.1.a)} \qquad \rho_1 \frac{\partial}{\partial t}\mathbf{u}_1^\varepsilon + (\mathbf{u}_1^\varepsilon, \nabla)\mathbf{u}_1^\varepsilon + \mathcal{E}_1^\varepsilon(\mathbf{u}_1^\varepsilon) + \nabla p_1^\varepsilon = \mathbf{f}_1 \quad \text{in } \Omega_1^\varepsilon,$$

$$\text{(1.1.b)} \qquad \text{div } \mathbf{u}_1^\varepsilon = 0,$$

$$\text{(1.2.a)} \qquad \mathbf{u}_1^\varepsilon = \mathbf{u}_2^\varepsilon, \quad p_1^\varepsilon = p_2^\varepsilon,$$

$$\text{(1.2.b)} \qquad \sum_{j=1}^n \sigma_{ij}(\mathbf{u}_1^\varepsilon) n_j = \sum_{j=1}^n \sigma_{ij}(\mathbf{u}_2^\varepsilon) n_j, \ \ 1 \le i \le n,$$

$$\text{(1.2.c)} \qquad (\mathbf{u}_1^\varepsilon \otimes \mathbf{u}_1^\varepsilon) \cdot \mathbf{n} = (\mathbf{u}_2^\varepsilon \otimes \mathbf{u}_2^\varepsilon) \cdot \mathbf{n} \quad \text{on } \Gamma_{12}^\varepsilon,$$

$$\text{(1.3.a)} \qquad \rho_2 \frac{\partial}{\partial t}\mathbf{u}_2^\varepsilon + (\mathbf{u}_2^\varepsilon, \nabla)\mathbf{u}_2^\varepsilon + \mathcal{E}_2^\varepsilon(\mathbf{u}_2^\varepsilon) + \nabla p_2^\varepsilon = \mathbf{f}_2 \quad \text{in } \Omega_2^\varepsilon,$$

$$\text{(1.1.b)} \qquad \text{div } \mathbf{u}_2^\varepsilon = 0.$$

REMARK 2.1. (i) The *global pressure* on Ω is given as

$$p^\varepsilon(x) = \chi_1\left(\frac{x}{\varepsilon}\right) p_1^\varepsilon(x) + \chi_2\left(\frac{x}{\varepsilon}\right) p_2^\varepsilon(x),$$

and the *global stress* is given as

$$\sigma_{ij}^\varepsilon(\mathbf{u}) = \chi_1\left(\frac{x}{\varepsilon}\right) \sigma_{ij}^1(\mathbf{u}_1) + \chi_2\left(\frac{x}{\varepsilon}\right) \sigma_{ij}^2(\mathbf{u}_2).$$

(ii) We need to supplement (1.1) with *boundary conditions* on $\partial\Omega$. Only those prescribed for p_1^ε and $\mathbf{u}^\varepsilon$ will survive the limit process as $\varepsilon \to 0$.
(iii) The equations (1.2) are just of the continuity of velocity fields, pressure, stress and inertia.
(iv) The components of (1.3.a) are given by

$$\rho_1 \frac{\partial}{\partial t} \, u_{2i}^\varepsilon + \sum_{j=1}^n u_{2j}^\varepsilon \, \frac{\partial}{\partial x_j} \, u_{2i}^\varepsilon - \sum_{k,l=1}^n \frac{\partial}{\partial x_i} \, a_{ijkl}\left(x, \frac{x}{\varepsilon}\right) \varepsilon_{kl}(\mathbf{u}_2^\varepsilon) + \frac{\partial p_2^\varepsilon}{\partial x_i} = f_i,$$

$i = 1, 2, \cdots, n$ and (1.1.a) is similar.

(v) We can allow the *quasi-static* cases $\rho_m = 0$ which are examples of *degenerate* evolution equations. Also we could modify the model to include a scaling by any positive power of ε for ρ_m, and then these would be lost in the limit. In this discussion we permit only an elliptic condition on ρ_m:

$$0 < c_3 \le \rho_m(x), \qquad x \in \Omega_m^\varepsilon$$

for some constant c_3.

A *variational (weak) formulation* of Navier-Stokes equations (1.1) $\sim$ (1.3) is to find

$$\mathbf{u}_m^\varepsilon \in L^2([0,T]; H^1(\Omega_m^\varepsilon)^n) \bigcap C_W([0,T]; L^2(\Omega_m^\varepsilon)^n), \qquad m = 1,2$$

with $p_1^\varepsilon = p_2^\varepsilon$ and $\mathbf{u}_1^\varepsilon = \mathbf{u}_2^\varepsilon$ on Γ_{12}^ε such that

$$\sum_{m=1}^{2} \left[\int_{\Omega_m^\varepsilon} \rho_m \frac{\partial}{\partial t} \mathbf{u}_m^\varepsilon \cdot \mathbf{v}_m + (\mathbf{u}_m^\varepsilon, \nabla)\mathbf{u}_m^\varepsilon \cdot \mathbf{v}_m dx + e_m^\varepsilon(\mathbf{u}_m, \mathbf{v}_m) \right]$$

$$= \sum_{m=1}^{2} \int_{\Omega_m^\varepsilon} \mathbf{f}_m \cdot \mathbf{v}_m dx \tag{2}$$

and

$$\sum_{m=1}^{2} \left[\int_{\Omega_m^\varepsilon} (\text{ div } \mathbf{u}_m^\varepsilon)\, \varphi_m dx \right] = 0 \tag{3}$$

for all divergence free vector fields $\mathbf{v}_m \in C^1([0,T]; H^1(\Omega_m^\varepsilon)^n)$, $m = 1,2$ with $\mathbf{v}_1 = \mathbf{v}_2$ on Γ_{12}^ε and all $\varphi_m \in C^1([0,T]; H^1(\Omega_m^\varepsilon))$, $m = 1,2$ with $\varphi_1 = \varphi_2$ on Γ_{12}^ε.
The space $C_W([0,T]; L^2(\Omega_m^\varepsilon)^n)$ is a subspace of $L^\infty([0,T]; L^2(\Omega_m^\varepsilon)^n)$ consisting of functions which are weakly continuous: $\int_{\Omega_m^\varepsilon} u(t,x) \cdot h(x)dx$ is a continuous function, for all $h \in L^2(\Omega_m^\varepsilon)^n$.

In the following discussion, the function space $\mathbf{V}_{div}$ is defined by

$$\mathbf{V}_{div} = \{\mathbf{v} \in \mathbf{V} : \text{div } \mathbf{v} = 0\}.$$

THEOREM 2.2. *For each* $\varepsilon > 0$ *and* $T > 0$, *there is a solution*

$$\mathbf{u}^\varepsilon \in L^2([0,T]; \mathbf{V}_{div}) \bigcap C_W([0,T]; L^2(\Omega)^n)$$

of the variational formulation (2) *and* (3) *for every given* $\mathbf{f} = \mathbf{f_0} + \text{div}\mathbf{F}$ *with*

$$\mathbf{f_0} \in L^1([0,T]; L^2(\Omega)^n), \qquad \mathbf{F} \in L^2([0,T]; L^2(\Omega)^{n^2})$$

and divergence free initial vector field $\mathbf{u}^\varepsilon(0) = \mathbf{u}_0^\varepsilon \in \mathbf{V}_{div}$.

Proof. The proof is originally discussed by J. Leray and there are several approaches to the existence of solutions, for example, see [3], [4], [6], [7], [12], [13]. All of these sources have used mainly the Galerkin approximation. □

We substitute $\mathbf{v}_1 = \mathbf{u}_1^\varepsilon$, $\mathbf{v}_2 = \mathbf{u}_2^\varepsilon$ into (2) to get

$$\sum_{m=1}^{2}\left[\frac{1}{2}\|\sqrt{\rho_m}\ \mathbf{u}_m^\varepsilon(t)\|^2_{L^2(\Omega_m^\varepsilon)^n} + \int_0^t e_m^\varepsilon(\mathbf{u}_m(\tau),\mathbf{u}_m(\tau))d\tau\right]$$

$$\leq \sum_{m=1}^{2}\left[\frac{1}{2}\|\sqrt{\rho_m}\ \mathbf{u}_m^\varepsilon(0)\|^2_{L^2(\Omega_m^\varepsilon)^n} + \int_0^t\int_{\Omega_m^\varepsilon}\mathbf{f}_m\cdot\mathbf{u}_m dxd\tau\right]$$

$$\text{(4)}\quad \leq \sum_{m=1}^{2}\left[\frac{1}{2}\|\sqrt{\rho_m}\ \mathbf{u}_m^\varepsilon(0)\|^2_{L^2(\Omega_m^\varepsilon)^n} + \int_0^t \|\mathbf{f}_m\|_{L^2(\Omega_m^\varepsilon)^n}\|\mathbf{u}_m\|_{L^2(\Omega_m^\varepsilon)^n}dxd\tau\right].$$

Applying Korn's inequality([9]) we see that

$$\|\mathbf{u}_1^\varepsilon\|_{L^2([0,T];L^2(\Omega_1^\varepsilon)^n)} \quad \textit{and} \quad \|\mathbf{u}_2^\varepsilon\|_{L^2([0,T];L^2(\Omega_2^\varepsilon)^n)}$$

are bounded. These facts, in turn, imply that

$$\|e_{kl}(\mathbf{u}_1^\varepsilon)\|_{L^2([0,T];L^2(\Omega_1^\varepsilon)^{n^2})}, \quad \|e_{kl}(\mathbf{u}_2^\varepsilon)\|_{L^2([0,T];L^2(\Omega_2^\varepsilon)^{n^2})},$$
$$\|\mathbf{u}_1^\varepsilon\|_{L^\infty([0,T];L^2(\Omega_1^\varepsilon)^n)}, \quad \|\mathbf{u}_2^\varepsilon\|_{L^\infty([0,T];L^2(\Omega_2^\varepsilon)^n)}$$

are bounded.

3. Two-Scale Convergence

We briefly review the two-scale convergence, and introduce the two-scale limit for the stain operator and prove a compactness theorem which is useful to deal with time dependent problems.

DEFINITION 3.1. *A sequence of functions* $\{u_\varepsilon\}$ *in* $L^2(\Omega)$ *is said to* two-scale converge *to a limit* $u_0(x,y) \in L^2(\Omega\times Y)$ *(we denote it by* $u_\varepsilon \xrightarrow{2} u_0(x,y)$*) if for any test function* $\psi(x,y)$ *in* $C_0^\infty(\Omega; C_\sharp^\infty(Y))$*, we have*

$$\int_\Omega u_\varepsilon(x)\psi(x,\frac{x}{\varepsilon})dx \longrightarrow \int_\Omega\int_Y u_0(x,y)\psi(x,y)dydx \qquad \text{as } \varepsilon\to 0.$$

This definition makes sense due to Nguetseng's compactness theorem, whose proof can be found in Allaire [1].

REMARK 3.1. *1.* (Nguetseng) *Any bounded sequence* $\{u_\varepsilon\}$ *in* $L^2(\Omega)$ *has two-scale convergent subsequence.*
2. Let $\{u_\varepsilon\}$ *be a bounded sequence in* $H^1(\Omega)$*. Then there are* $u_0 \in H^1(\Omega)$ *and* $U(x,y) \in L^2(\Omega; H^1_\sharp(Y))$ *such that, up to a subsequence,* $\{u_\varepsilon\}$ *two-scale converges to* $u_0(x)$ *and* $\{\nabla u_\varepsilon\}$ *two-scale converges to* $\nabla u_0(x) + \nabla_y U(x,y)$.
3. Suppose a sequence of vector fields $\{\mathbf{u}_\varepsilon\}$ *and* $\{\text{div } \mathbf{u}_\varepsilon\}$ *are bounded in* $L^2(\Omega)^n$ *and* $L^2(\Omega)$*, respectively. Then there are* $\mathbf{u}_0 \in H^1(\Omega)^n$ *and* $\mathbf{U}(x,y) \in L^2(\Omega; H^1_\sharp(Y))^n$ *such that, up to a subsequence,* $\{\mathbf{u}_\varepsilon\}$ *two-scale converges to* $\mathbf{u}_0$ *and* $\{\text{div } \mathbf{u}_\varepsilon\}$ *two-scale converges to* $\text{div } \mathbf{u}_0 + \text{div}_y \, \mathbf{U}(x,y)$.

We present two-scale strain behavior of bounded vector fields for which its strain is bounded. We denote the strain operator by $\mathbf{e}(\mathbf{u})$, that is, the i-j component $\mathbf{e}(\mathbf{u})_{ij}$ of $\mathbf{e}(\mathbf{u})$ is $\mathbf{e}(\mathbf{u})_{ij} = \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial \mathbf{u}_i}{\partial x_j} + \frac{\partial \mathbf{u}_j}{\partial x_i}\right)$, and also $\mathbf{e}(\mathbf{u})_y = \left(\frac{1}{2}\left(\frac{\partial \mathbf{u}_i}{\partial y_j} + \frac{\partial \mathbf{u}_j}{\partial y_i}\right)\right)$.

PROPOSITION 3.1. *Suppose that a sequence of vector fields* $\{\mathbf{u}_\varepsilon\}$ *is bounded in* $L^2(\Omega)^n$ *and the sequence of matrices* $\{\mathbf{e}(\mathbf{u}_\varepsilon)\}$ *is bounded in* $L^2(\Omega)^{n\times n}$*, respectively. Then there is a subsequence* $\{\mathbf{u}_{\varepsilon_\mathbf{j}}\}$ *of* $\{\mathbf{u}_\varepsilon\}$ *such that* $\{\mathbf{u}_{\varepsilon_\mathbf{j}}\}$ *two-scale converges to* $\mathbf{u}$ *and* $\mathbf{e}(\mathbf{u}_{\varepsilon_j})$ *two-scale converges to* $\mathbf{e}(\mathbf{u}_0) + \mathbf{e}_y(\mathbf{U})$*, for some* $\mathbf{u}(x) \in L^2(\Omega)^n$ *and* $\mathbf{U} \in L^2(\Omega; H^1_\sharp(Y))^n$.

Proof. It can be proved by Korn's inequality and Remark 3.1-2. (For a complete discussion about Korn's inequality, we refer [9]). □

In order to deal with the homogenized property for parabolic equations, we prove the following compactness theorem which generalizes the compactness theorem for elliptic problems described in Remark 3.1.

THEOREM 3.2. *Let* $0 < T \le \infty$*. Suppose that* $\{u_\varepsilon\}$ *is a bounded sequence of forms in* $L^2([0,T); L^2(\Omega))$*. Then there are a subsequence* $\{u_{\varepsilon_j}\}$ *of* $\{u_\varepsilon\}$ *and* $U_0(t,x,y) \in L^2\big([0,T); L^2(\Omega \times Y)\big)$ *such that for any* $\psi(t,x,y) \in L^2\left([0,T); C_0^\infty(\Omega; C_\sharp^\infty(Y))\right)$

$$\lim_{\varepsilon_j \to 0} \int_0^T \int_\Omega u_{\varepsilon_j}(t,x)\, \psi\left(t, x, \frac{x}{\varepsilon_j}\right) dxdt = \int_0^T \int_{\Omega\times Y} U_0(t,x,y)\, \psi(t,x,y) dxdydt.$$

Proof. We have a given bounded sequence $\{u_\varepsilon\}$ in $L^2([0,T); L^2(\Omega))$. There is a positive constant C such that $\|u_\varepsilon\|_{L^2([0,T);L^2(\Omega))} \le C$. Then let $\mathcal{F}_\varepsilon(\Psi) \equiv \int_0^T \int_\Omega u_\varepsilon(t,x)\Psi(t,x,\frac{x}{\varepsilon})dxdt$ and $\mathcal{D} \equiv L^2(\Omega; C_\sharp(Y))$. We notice that

$$
\begin{aligned}
&| \mathcal{F}_\varepsilon(\Psi) | \leq \int_0^T \|u_\varepsilon(t)\|_{L^2(\Omega)} \left\|\Psi\left(t, x, \frac{x}{\varepsilon}\right)\right\|_{L^2(\Omega)} dt \\
&\leq C \left[\int_0^T \left(\max_{y\in Y} \| \Psi(t,x,y) \|_{L^2(\Omega)}\right)^2 dt\right]^{\frac{1}{2}} = C\left\{\int_0^T \| \Psi(t) \|_{\mathcal{D}}^2 \, dt\right\}^{\frac{1}{2}}
\end{aligned}
$$

for every $\Psi \in L^2([0,T);\mathcal{D})$. This leads to $\mathcal{F}_\varepsilon \in \big(L^2([0,T);\mathcal{D})\big)'$. So a subsequence $\{\mathcal{F}_{\varepsilon_j}\}$ is *weak**-convergent to some $\mathcal{U}_0 \in \big(L^2([0,T);\mathcal{D})\big)'$. Therefore from the fact that for $\Psi \in L^2([0,T);\mathcal{D})$,

$$
\left\|\Psi\left(t, x, \frac{x}{\varepsilon_j}\right)\right\|_{L^2(\Omega)} \leq \| \Psi(t) \|_{\mathcal{D}}, \qquad t \in [0,T)
$$

together with Lebesgue Dominated Convergence Theorem, we have

$$
\begin{aligned}
&| \mathcal{U}_0(\Psi) | = \lim_{\varepsilon_j\to 0} | \mathcal{F}_{\varepsilon_j}(\Psi) | \\
&\leq \limsup_{\varepsilon_j\to 0} \int_0^T \| u^{\varepsilon_j}(t) \|_{L^2(\Omega)} \left\|\Psi\left(t, x, \frac{x}{\varepsilon_j}\right)\right\|_{L^2(\Omega)} dt \\
&\leq C \left[\int_0^T \limsup_{\varepsilon_j\to 0} \left\|\Psi\left(t, x, \frac{x}{\varepsilon_j}\right)\right\|_{L^2(\Omega)}^2 dt\right]^{\frac{1}{2}} = C\left[\int_0^T \| \Psi(t) \|_{L^2(\Omega\times Y)}^2 \, dt\right]^{\frac{1}{2}}.
\end{aligned}
$$

Since $L^2([0,T);\mathcal{D})$ is dense in $L^2([0,T);L^2(\Omega\times Y)) = L^2([0,T)\times\Omega\times Y)$, it follows that $\mathcal{U}_0$ is in $L^2([0,T)\times\Omega\times Y)'$. By Riesz Representation Theorem, we have

$$
\mathcal{U}_0(\Psi) = \left[\int_0^T \langle U_0(t), \Psi(t)\rangle_{L^2(\Omega\times Y)}^2 \, dt\right]^{\frac{1}{2}}, \quad \Psi \in L^2([0,T)\times\Omega\times Y)
$$

for some $U_0 \in L^2\big([0,T); L^2(\Omega\times Y)\big)$. Therefore we get

$$
\begin{aligned}
\lim_{\varepsilon_j\to 0} \int_0^T \int_\Omega u^{\varepsilon_j}(t,x)\Psi\left(t, x, \frac{x}{\varepsilon_j}\right) dxdt &\equiv \lim_{\varepsilon_j\to 0} \mathcal{F}_{\varepsilon_j}(\Psi) = \mathcal{U}_0(\Psi) \\
&= \int_0^T \int_Y \int_\Omega U_0(t,x,y)\Psi(t,x,y)dxdydt.
\end{aligned}
$$

□

4. Homogenized Navier-Stokes system

We will consider the limit equations for Navier-Stokes system.

Let us denote the scaled characteristic functions by

$$\chi_m^\varepsilon \equiv \chi_m\left(\frac{x}{\varepsilon}\right),\ m=1,2.$$

Firstly, Remark 3.1, Proposition 3.1 and a priori estimates yield the existence of subsequences (it is also denoted by $\mathbf{u}^\varepsilon$) which two-scale converge to some

$$\mathbf{u}\in L^\infty([0,T];H^1(\Omega)^n),\quad \mathbf{U}\in L^\infty([0,T];L^2(\Omega,H^1_\sharp(Y)^n))$$

as follows:

$$\mathbf{u}^\varepsilon(t,x)\xrightarrow{2}\mathbf{u}(t,x),\qquad e_{kl}(\mathbf{u}^\varepsilon)\xrightarrow{2}e_{kl}(\mathbf{u})+e^y_{kl}(\mathbf{U}(x,y)),$$

wherein $e^y_{kl}(\mathbf{U})\equiv\frac{1}{2}(\frac{\partial\mathbf{U}_l}{\partial y_k}+\frac{\partial\mathbf{U}_k}{\partial y_l})$. Secondly, by virtue of Korn's inequality, we get that

$$\|\mathbf{u}_1^\varepsilon\|_{L^\infty([0,T];H^1(\Omega_1^\varepsilon)^n)},\qquad \|\mathbf{u}_2^\varepsilon\|_{L^\infty([0,T];H^1(\Omega_2^\varepsilon)^n)}$$

are bounded, so then Rellich Theorem yields $\mathbf{u}^\varepsilon(t)$ *converges strongly* $\mathbf{u}(t)$ in $L^2(\Omega)^n$, for each $t\in[0,T]$. Thirdly, the divergence free condition of velocity fields $\mathbf{u}_m^\varepsilon$ yields;

$$\text{div}\ \mathbf{u}(x)+\text{div}_y\ \mathbf{U}(x,y)=0 \tag{5}$$

on Ω(Remark 3.1-3). We take the weak limit on the condition (3) to get div $\mathbf{u}=0$. Therefore, from (5), we have $\text{div}_y\ \mathbf{U}(x,y)=0$.

The ε-equation (2) can be displayed as;

$$\begin{aligned}&\int_\Omega \chi_1^\varepsilon(x)\left\{\rho_1(x)\ \frac{\partial}{\partial t}\mathbf{u}_1^\varepsilon(x)\ \cdot\mathbf{v}_1^\varepsilon(x)+(\mathbf{u}_1^\varepsilon(x),\nabla)\mathbf{u}_1^\varepsilon(x)\cdot\mathbf{v}_1^\varepsilon(x)\right\}dx\\ &+\int_\Omega\left\{\chi_2^\varepsilon(x)\rho_2(x)\ \frac{\partial}{\partial t}\mathbf{u}_2^\varepsilon(x)\ \cdot\mathbf{v}_2^\varepsilon(x)+\chi_2^\varepsilon(x)(\mathbf{u}_2^\varepsilon(x),\nabla)\mathbf{u}_2^\varepsilon(x)\cdot\mathbf{v}_2^\varepsilon(x)\right\}dx\\ &\qquad+e_1^\varepsilon(\mathbf{u}_1,\mathbf{v}_1^\varepsilon)+e_2^\varepsilon(\mathbf{u}_2,\mathbf{v}_2^\varepsilon)\\ &=\int_\Omega\chi_1^\varepsilon(x)\mathbf{f}_1(x)\cdot\mathbf{v}_1^\varepsilon(x)dx+\int_\Omega\chi_2^\varepsilon(x)\mathbf{f}_2(x)\cdot\mathbf{v}_2^\varepsilon(x)dx,\end{aligned}\tag{6}$$

for all divergence free fields $\mathbf{v}^\varepsilon_m \in C^1([0,T];H^1(\Omega^\varepsilon_m)^n)$, $m=1,2$ with $\mathbf{v}_1=\mathbf{v}_2$ on Γ^ε_{12} and $\mathbf{v}_1(T)=\mathbf{v}_2(T)=0$. We rewrite (6) as;

$$\begin{aligned}
&\int_0^T\int_\Omega \chi^\varepsilon_1(x)\rho_1(x)\,\mathbf{u}^\varepsilon_1(t,x)\cdot\frac{\partial}{\partial t}\mathbf{v}^\varepsilon_1(t,x)dxdt+\int_0^T e^\varepsilon_1(\mathbf{u}_1(t),\mathbf{v}^\varepsilon_1(t))dt\\
&\qquad+\int_0^T\int_\Omega \chi^\varepsilon_1(x)\left\{\mathbf{u}^\varepsilon_1(t,x)\otimes\mathbf{u}^\varepsilon_1(t,x)\right\}\cdot\nabla\mathbf{v}^\varepsilon_1(t,x)dxdt\\
&\quad+\int_0^T\int_\Omega \chi^\varepsilon_2(x)\rho_2(x)\,\mathbf{u}^\varepsilon_2(t,x)\cdot\frac{\partial}{\partial t}\mathbf{v}^\varepsilon_2(t,x)dxdt+\int_0^T e^\varepsilon_2(\mathbf{u}_2,\mathbf{v}^\varepsilon_2)dt\\
&\qquad+\int_0^T\int_\Omega \chi^\varepsilon_2(x)\left\{\mathbf{u}^\varepsilon_2(t,x)\otimes\mathbf{u}^\varepsilon_2(t,x)\right\}\cdot\nabla\mathbf{v}^\varepsilon_2(t,x)dxdt\\
&=\int_{\Omega^\varepsilon_1}\rho_1(x)\,\mathbf{u}^\varepsilon_1(0,x)\cdot\mathbf{v}^\varepsilon_1(0,x)dx+\int_{\Omega^\varepsilon_2}\rho_2(x)\,\mathbf{u}^\varepsilon_2(0,x)\cdot\mathbf{v}^\varepsilon_2(0,x)dxdt\\
&+\int_0^T\int_\Omega \chi^\varepsilon_1(x)\mathbf{f}_1(t,x)\cdot\mathbf{v}^\varepsilon_1(t,x)dxdt+\int_0^T\int_\Omega \chi^\varepsilon_2(x)\mathbf{f}_2(t,x)\cdot\mathbf{v}^\varepsilon_2(t,x)dxdt.
\end{aligned}\tag{7}$$

Applying any divergence free test field $\mathbf{v}\in H^1(\Omega)^n$ in the place of $\mathbf{v}_1$ and $\mathbf{v}_2$, we have the (weak) limit of ε-equation (7);

$$\begin{aligned}
&\int_0^T\int_\Omega\left\{\rho(x)\,\frac{\partial\mathbf{u}}{\partial t}(t,x)\cdot\mathbf{v}(t,x)+\left\{\mathbf{u}(t,x)\otimes\mathbf{u}(t,x)\right\}\cdot\nabla\mathbf{v}(t,x)\right\}dxdt\\
&\quad+\int_0^T e(\mathbf{u}(t)+\mathbf{U}(t),\mathbf{v}(t))dt=\int_0^T\int_\Omega \mathbf{f}(t,x)\cdot\mathbf{v}(t,x)dxdt
\end{aligned}\tag{8}$$

with the *effective coefficients* given by

$$\rho(x)=|Y_1|\rho_1(x)+|Y_2|\rho_2(x),\quad \mathbf{f}(x)=|Y_1|\mathbf{f}_1(x)+|Y_2|\mathbf{f}_2(x)\tag{9}$$

and the homogenized elasticity bilinear form is defined by;

$$e(\mathbf{u}+\mathbf{U},\mathbf{v})\equiv\sum_{i,j,k,l=1}^n\int_Y\int_\Omega a_{ijkl}(x,y)\left\{\varepsilon_{kl}(\mathbf{u})+e^y_{kl}(\mathbf{U}(x,y))\right\}\varepsilon_{ij}(\mathbf{v})dxdy,$$

where the corresponding effective elasticity tensor is

$$a_{ijkl}(x,y)=\chi_1(y)a^1_{ijkl}(x,y)+\chi_2(y)a^2_{ijkl}(x,y).$$

In order to investigate the cell problem of the limit system, we apply divergence free test fields $\varepsilon\mathbf{V}\left(x,\frac{x}{\varepsilon}\right)$, where $\mathbf{V}(x,y)\in C^\infty_0(\Omega;C^\infty_\sharp(Y)^n)$ in the place of $\mathbf{v}_m$ on (6) to have its two-scale limit;

$$\sum_{i,j,k,l=1}^n\int_\Omega\int_Y a_{ijkl}(x,y)[\varepsilon_{kl}(\mathbf{u}(x))+\varepsilon^y_{kl}(\mathbf{U}(x,y))]\varepsilon^y_{ij}(\mathbf{V}(x,y))dydx=0.\tag{10}$$

Indeed, we can note the two-scale limit of the nonlinear term as follows;

$$\int_{\Omega_1^\varepsilon} (\mathbf{u}_1^\varepsilon(x), \nabla)\mathbf{u}_1^\varepsilon(x) \cdot \varepsilon \mathbf{V}\left(x, \frac{x}{\varepsilon}\right) dx + \int_{\Omega_2^\varepsilon} (\mathbf{u}_2^\varepsilon(x), \nabla)\mathbf{u}_2^\varepsilon(x) \cdot \varepsilon \mathbf{V}\left(x, \frac{x}{\varepsilon}\right) dx$$

$$= \sum_{m=1}^{2} \int_{\Omega_m^\varepsilon} \{\mathbf{u}_m^\varepsilon(x) \otimes \mathbf{u}_m^\varepsilon(x)\} \cdot \left\{\varepsilon \nabla_x \mathbf{V}\left(x, \frac{x}{\varepsilon}\right) + \nabla_y \mathbf{V}\left(x, \frac{x}{\varepsilon}\right)\right\} dx$$

$$\xrightarrow{2} \int_{\Omega \times Y} \mathrm{div}_y \{\mathbf{u}(x) \otimes \mathbf{u}(x)\} \cdot \mathbf{V}(x, y)\, dxdy = 0.$$

By density, (10) holds for all divergence free fields $\mathbf{V} \in L^2(\Omega; H^1_\sharp(Y)^n)$. Therefore, the corresponding strong formulation of (10) is;

$$(11) \quad -\sum_{j=1}^{n} \sum_{k,l=1}^{n} \frac{\partial}{\partial y_j}\, a_{ijkl}(x,y)\, \left[\varepsilon_{kl}(\mathbf{u}) + \varepsilon^y_{kl}(\mathbf{U}(x,y))\right] + \frac{\partial}{\partial y_i} q = 0,$$

$1 \le i \le n$, for some scalar function $q(x)$. This leads to the

Cell Problem. Find $\mathbf{X}(y) \in H^1_\sharp(Y)^{n^3}$, $\mathbf{\Pi}(y) \in H^1_\sharp(Y)^{n^2}$ such that

$$(12) \quad \begin{cases} -\displaystyle\sum_{j,k,l=1}^{n} \frac{\partial}{\partial y_j}\, a_{ijkl}(x,\cdot)\, \left[\delta^{rs}_{kl} + \varepsilon^y_{kl}(\mathbf{X}_{rs})\right] \\ \qquad\qquad + \dfrac{\partial}{\partial y_i} \mathbf{\Pi}_{rs} = 0, \quad 1 \le i, r, s \le n, \quad x \in \Omega \\ \displaystyle\sum_{m=1}^{n} \frac{\partial}{\partial y_m}\, \mathbf{X}^m_{rs} = 0, \qquad\qquad 1 \le r, s \le n, \end{cases}$$

where we set

$$\delta^{rs}_{kl} = \begin{cases} 1 & \text{when } r = k \text{ and } s = l \\ 0 & \text{otherwise.} \end{cases}$$

The above cell problem is *nonhomogeneous Stokes equations.* In fact, the equation (12) can be rewritten as; for each $1 \le r, s \le n$

$$-\sum_{j,k,l=1}^{n} \frac{\partial}{\partial y_j}\, a_{ijkl}(x,\cdot)\varepsilon^y_{kl}(\mathbf{X}_{rs}) + \frac{\partial}{\partial y_i}\mathbf{\Pi}_{rs} = \sum_{j=1}^{n} \frac{\partial}{\partial y_j}\, a_{ijrs}(x,\cdot), \quad 1 \le i \le n.$$

We notice that $\mathbf{X}_{rs}$ and $\mathbf{\Pi}_{rs}$ are symmetric; that is,

$$\mathbf{X}_{rs} = \mathbf{X}_{sr}, \qquad \mathbf{\Pi}_{rs} = \mathbf{\Pi}_{sr},$$

for all r, s. Now, we let

$$\mathbf{U} \equiv \sum_{r,s=1}^{n} \mathbf{X}_{rs}\varepsilon_{rs}(\mathbf{u}) \qquad \text{and} \qquad q \equiv \sum_{r,s=1}^{n} \mathbf{\Pi}_{rs}\varepsilon_{rs}(\mathbf{u})$$

to find $\mathbf{U}$, q that satisfy the cell equation (11). The following observation:

$$e(\mathbf{u}+\mathbf{U},\mathbf{v}) = \sum_{i,j,k,l=1}^{n} \int_Y\int_\Omega a_{ijkl}(x,y)\left\{\varepsilon_{kl}(\mathbf{u})+ e^y_{kl}(\mathbf{U}(x,y))\right\}\varepsilon_{ij}(\mathbf{v})dxdy$$

$$= \sum_{i,j,k,l=1}^{n} \int_\Omega \left\{ \sum_{r,s=1}^{n} \int_Y a_{ijrs}(x,y)\left[\delta^{kl}_{rs} + \varepsilon^y_{rs}(\mathbf{X}_{kl})\right] dy \right\} \varepsilon_{kl}(\mathbf{u})\, \varepsilon_{ij}(\mathbf{v})dx.$$

allows us to define the *homogenized effective elasticity tensor*

$$b_{ijkl}(x) \equiv \sum_{r,s=1}^{n} \int_Y a_{ijrs}(x,y)\left[\delta^{kl}_{rs} + \varepsilon^y_{rs}(\mathbf{X}_{kl})\right] dy, \tag{13}$$

and the *homogenized effective elasticity operator* $\mathcal{E}$

$$\mathcal{E}(\mathbf{u})_i = -\sum_{j=1}^{n}\sum_{k,l=1}^{n} \partial_j\, b_{ijkl}(x)\, \varepsilon_{kl}(\mathbf{u}), \quad 1 \le i \le n, \tag{14}$$

whenever $\mathbf{u} \in \mathbf{V}$. Therefore, the strong formulation of (8) is

$$\rho\frac{\partial}{\partial t}\mathbf{u} + (\mathbf{u},\nabla)\mathbf{u} + \mathcal{E}(\mathbf{u}) + \nabla p = \mathbf{f},$$

for some potential function $p(t,x)$.

We summarize the homogenizing process as follows;

THEOREM 4.1. *The homogenizing process for the Navier-Stokes systems (1.1) $\sim$ (1.3) gives rise to the same type of Navier-Stokes systems displayed as*

$$\rho\frac{\partial}{\partial t}\mathbf{u} + (\mathbf{u},\nabla)\mathbf{u} + \mathcal{E}(\mathbf{u}) + \nabla p = \mathbf{f},$$
$$\operatorname{div}\mathbf{u} = 0,$$

with the effective coefficients (9), (13) and (14). It also produces the cell problems (12) which are the classical Stokes equations of matrix-type on torus.

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