JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **33**, No. 1, February 2020
http://dx.doi.org/10.14403/jcms.2020.33.1.65

THE m-STEP COMPETITION GRAPHS OF d-PARTIAL ORDERS

Jihoon Choi*

Abstract. The notion of m-step competition graph was introduced by Cho *et al.* in 2000 as an interesting variation of competition graph. In this paper, we study the m-step competition graphs of d-partial orders, which generalizes the results obtained by Park *et al.* in 2011 and Choi *et al.* in 2018.

1. Introduction

In this paper, all the graphs and digraphs are assumed to be finite and simple unless otherwise stated. We write $u \to v$ for an arc (u, v) in a digraph.

The *competition graph* of a given digraph D, denoted by $C(D)$, is defined to be the graph such that $V(C(D)) = V(D)$ and $E(C(D)) = \{xy \mid (x, z), (y, z) \in A(D)$ for some $z \in A(D)\}$. Since its introduction, a lot of variations of competition graph have been introduced and studied (see [1, 2, 8, 9, 10, 13] for reference). One example is the m-step competition graph, which was introduced by Cho *et al.* [4]. Let D be a digraph and m be a positive integer. A vertex y is called an *m-step prey* of a vertex x in D if there is a directed walk from x to y of length m. The *m-step competition graph* of D, denoted by $C^m(D)$, is defined to be the graph such that $V(C^m(D)) = V(D)$ and xy is an edge in $C^m(D)$ if and only if there exists an m-step common prey of u and v in D. The readers may refer to [4, 9, 11] for the structural properties of m-step competition graphs, [1, 8, 13] for the characterizations of paths and cycles which are

Received January 07, 2020; Accepted January 17, 2020.
2010 Mathematics Subject Classification: Primary 05C20; Secondary 05C75.
Key words and phrases: competition graph, m-step competition graph, d-partial order, partial order m-step competition dimension, partial order competition exponent.
*This work was supported by the research grant of Cheongju University (2018.03.01. – 2020.02.29.)

realizable as the m-step competition graph, and [2, 7, 10, 12] for the matrix sequence $\{C^m(D)\}_{m=1}^{\infty}$.

Let d be a positive integer. For $x = (x_1, x_2, \ldots, x_d)$, $y = (y_1, y_2, \ldots, y_d) \in \mathbb{R}^d$, we write $x \prec y$ if $x_i < y_i$ for each $i = 1, \ldots, d$. If $x \prec y$ or $y \prec x$, then we say that x and y are *comparable* in $\mathbb{R}^d$. Otherwise, we say that x and y are *incomparable* in $\mathbb{R}^d$. For a finite subset S of $\mathbb{R}^d$, let D_S denote the digraph defined by $V(D_S) = S$ and $A(D_S) = \{(x, v) \mid v, x \in S, v \prec x\}$. A digraph is called a *d-partial order* if $D = D_S$ for a finite subset S of $\mathbb{R}^d$. It is clear that every d-partial order is transitive, and therefore acyclic.

A 2-partial order is called a *doubly partial order*, which was introduced by Cho and Kim [3]. They proved that the interval graphs are exactly the graphs with "partial order competition dimensions" at most two, by showing that every competition graph of a doubly partial order is an interval graph, and that every interval graph together with some additional isolated vertices is the competition graph of a doubly partial order. Park *el al.* [11] characterized the graphs which can be represented as the m-step competition graphs of doubly partial orders by adding sufficiently many isolated vertices. In this paper, we study the m-step competition graphs of d-partial orders, which generalizes the results obtained by Park *et al.*

2. A characterization of the m-step competition graphs of d-partial orders

Let $\mathbf{1}$ denote the all-one vector $(1, 1, \ldots, 1)$ in $\mathbb{R}^d$. For $\mathbf{x} \in \mathbb{R}^d$, the dot product of $\mathbf{x}$ and $\mathbf{1}$ is defined by $\mathbf{x} \cdot \mathbf{1} = \sum_{i=1}^{d} x_i$. Let

$$\mathcal{H}^d = \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{x} \cdot \mathbf{1} = 0\}, \qquad \mathcal{H}^d_+ := \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{x} \cdot \mathbf{1} > 0\}.$$

For a point $\mathbf{p}$ in $\mathcal{H}^d_+$, let $\triangle^{d-1}(\mathbf{p})$ be the intersection of the closed cone $\{\mathbf{x} \in \mathbb{R}^d \mid x_i \leq p_i \ (i = 1, \ldots, d)\}$ and the hyperplane $\mathcal{H}^d$. For a subset A of $\mathbb{R}^d$, $\text{int}(A)$ denotes the interior of A with respect toe the standard topology in $\mathbb{R}^d$. Then the following are true.

LEMMA 2.1 ([6]). *For $\mathbf{p} \in \mathcal{H}^d_+$, the set $\triangle^{d-1}(\mathbf{p})$ is a regular $(d-1)$-simplex.*

PROPOSITION 2.2 ([6]). *For $\mathbf{p}, \mathbf{q} \in \mathcal{H}^d_+$, $\triangle^{d-1}(\mathbf{p}) \subset int(\triangle^{d-1}(\mathbf{q}))$ if and only if $\mathbf{p} \prec \mathbf{q}$.*

Two geometric figures in $\mathbb{R}^d$ are said to be *homothetic* if one can be mapped into the other by dilation and translation. Then the following is true.

PROPOSITION 2.3 ([6]). *For* $\mathbf{p}, \mathbf{q} \in \mathcal{H}^d_+$, $\triangle^{d-1}(\mathbf{p})$ *and* $\triangle^{d-1}(\mathbf{q})$ *are homothetic.*

Let $\mathcal{F}^{d-1}$ denote the set of regular $(d-1)$-simplices in $\mathbb{R}^d$ contained in $\mathcal{H}^d$ and homothetic to $\triangle^{d-1}(\mathbf{1})$. Then there is a one to one correspondence between $\mathcal{H}^d_+$ and $\mathcal{F}_{d-1}$.

COROLLARY 2.4 ([6]). *For each integer* $d \geq 2$, *the function* $\triangle^{d-1} : \mathcal{H}^d_+ \to \mathcal{F}^{d-1}$ *mapping* $\mathbf{p}$ *to* $\triangle^{d-1}(\mathbf{p})$ *is bijective.*

As an analogue of Theorem 2.9 in [6], we characterize the m-step competition graph of a d-partial order as follows.

THEOREM 2.5. *Let* m *and* d *be positive integers. Then a graph* G *is the* m*-step competition graph of a* d*-partial order if and only if there exist a subset* $\mathcal{F}$ *of* $\mathcal{F}^{d-1}$ *and a bijection* $f : V(G) \to \mathcal{F}$ *such that*

($\star$) *two vertices* v *and* w *are adjacent in* G *if and only if there exist two sequences* $(v_0, v_1, \ldots, v_m)$ *and* $(w_0, w_1, \ldots, w_m)$ *on* $V(G)$ *such that* $v_0 = v$, $w_0 = w$, $v_m = w_m$, *and for each* $i = 1, 2, \ldots, m$, $f(v_i) \subset \text{int}(f(v_{i-1}))$ *and* $f(w_i) \subset \text{int}(f(w_{i-1}))$.

Proof. ($\Rightarrow$) Assume that G is the m-step competition graph of some d-partial order D. For a positive real number k which is large enough, we translate all the vertices of D by $T : v \mapsto v + k\mathbf{1}$ so that we may assume $V(D) \subset \mathcal{H}^d_+$. Let $\mathcal{F} = \{\triangle^{d-1}(v) \mid v \in V(D)\}$. Then $\mathcal{F} \subset \mathcal{F}^{d-1}$. Let $f : V(G) \to \mathcal{F}$ be the function defined by $f(v) = \triangle^{d-1}(v)$. Then f is a bijection by Corollary 2.4. The property ($\star$) immediately follows from the definition of m-step competition graph and Proposition 2.2.

($\Leftarrow$) Assume there exist $\mathcal{F} \subset \mathcal{F}^{d-1}$ and a bijection $f : V(G) \to \mathcal{F}$ such that the property ($\star$) is true. By Corollary 2.4, each element in $\mathcal{F}$ can be written in the form of $\triangle^{d-1}(\mathbf{p})$ for some $\mathbf{p} \in \mathcal{H}^d_+$. Take two vertices v and w in G. Then, by the property ($\star$) and Proposition 2.2, v and w are adjacent in G if and only if v and w have an m-step common prey in the d-partial order D_S where $S = \{\mathbf{p} \in \mathbb{R}^d \mid \triangle^{d-1}(\mathbf{p}) \in \mathcal{F}\}$. Thus $G = C^m(D_S)$. □

3. Partial order m-step competition dimensions of graphs

We denote by $\mathbb{Z}_{>0}$ and $\mathbb{Z}_{\geq 0}$ the set of positive integers and the set of nonnegative integers, respectively. In addition, I_k denotes the set of k isolated vertices for each $k \in \mathbb{Z}_{\geq 0}$.

To study the competition graphs of d-partial orders, Choi *et al.* [6] introduced the notion of the partial order competition dimension of a graph.

Definition 3.1 ([6]). The *partial order competition dimension* of a graph G, denoted by $\dim_{\text{poc}}(G)$, is defined to be the smallest positive integer d such that G together with k additional isolated vertices is the competition graph of some d-partial order D for some $k \in \mathbb{Z}_{\geq 0}$, i.e.,

$$\dim_{\text{poc}}(G) := \min\{d \in \mathbb{Z}_{>0} \mid \exists k \in \mathbb{Z}_{\geq 0}, \exists S \subset \mathbb{R}^d \text{ s.t. } G \cup I_k = C(D_S)\}.$$

In this section, we introduce the notion of partial order m-step competition dimension of a graph to generalize that of partial order competition dimension and investigate basic properties of m-step competition graphs of d-partial orders in terms of it.

Lemma 3.2. *For a transitive digraph D and a positive integer m, every m-step prey of x in D is a k-step prey of x for each $k = 1, \ldots, m$.*

Proof. It easily follows from the transitivity of D. □

Lemma 3.3. *Every d-partial order is isomorphic to a $(d+1)$-partial order.*

Proof. We mimic the proof of Proposition 3.1 in [6]. Let D be a d-partial order. For each $\mathbf{v} = (v_1, \ldots, v_d) \in V(D) \subset \mathbb{R}^d$, we define $\tilde{\mathbf{v}} \in \mathbb{R}^{d+1}$ by $\tilde{\mathbf{v}} = \left(v_1, \ldots, v_d, \sum_{i=1}^d v_i\right)$. Let $\tilde{V} = \{\tilde{\mathbf{v}} \mid \mathbf{v} \in V(D)\}$. Then $D_{\tilde{V}}$ is a $(d+1)$-partial order. Take $\mathbf{v} = (v_1, \ldots, v_d)$ and $\mathbf{w} = (w_1, \ldots, w_d)$ in D. Then

$$\begin{aligned} \tilde{\mathbf{v}} \prec \tilde{\mathbf{w}} \quad &\Leftrightarrow \quad v_i < w_i \ (i = 1, \ldots, d) \ \text{ and } \ \sum_{i=1}^d v_i < \sum_{i=1}^d w_i \\ &\Leftrightarrow \quad v_i < w_i \ (i = 1, \ldots, d) \\ &\Leftrightarrow \quad \mathbf{v} \prec \mathbf{w}, \end{aligned}$$

and therefore D is isomorphic to $\tilde{D}$. □

The following proposition is an immediate consequence of Lemma 3.3.

PROPOSITION 3.4 ([6]). *For positive integers m and d, the m-step competition graph of a d-partial order is (isomorphic to) the m-step competition graph of a $(d+1)$-partial order.*

Let G be a graph. A *clique* of G is a vertex subset in which all the vertices are pairwise adjacent in G. For a clique K and an edge e of G, we say that K *covers* e if K contains the two end vertices of e. An *edge clique cover* of G is a family of cliques of G which cover all the edges of G. The minimum cardinality of an edge clique cover of G is called the *edge clique cover number* of G and denoted by $\theta_e(G)$.

THEOREM 3.5. *Let G be a graph and m be a positive integer. Then there exist a positive integer d and a nonnegative integer k such that G together with k additional isolated vertices is the m-step competition graph of some d-partial order.*

Proof. Let $v_1, \dots, v_n$ be the vertices of D. We define a map $\phi : V(D) \to \mathbb{R}^n$ so that the jth coordinate of $\phi(v_i)$ $(i = 1, \dots, n)$ is given by

$$\phi(v_i)_j = \begin{cases} 2 & \text{if } j = i; \\ 4 & \text{if } j \neq i. \end{cases}$$

Let $\theta = \theta_e(G)$ and $\mathcal{C} = \{C_1, C_2, \dots, C_\theta\}$ be an edge clique cover of G consisting of maximal cliques. For each $t \in \{1, 2, \dots, m\}$, we define a map $\psi_t : \mathcal{C} \to \mathbb{R}^n$ so that the jth coordinate of $\psi_t(C_l)$ $(l = 1, \dots, \theta)$ is given by

$$(\psi_t(C_l))_j = \begin{cases} 1 - \frac{t}{m+1} & \text{if } v_j \in C_l; \\ 3 - \frac{t}{m+1} & \text{if } v_j \notin C_l. \end{cases}$$

Let $V = \{\phi(v_i) \mid i = 1, 2, \dots, n\} \cup \{\psi_t(C_l) \mid t = 1, 2, \dots, m,\ l = 1, 2, \dots, \theta\} \subseteq \mathbb{R}^n$. Then, in the d-partial order D_V, it easily be checked that the vertex $\psi_t(C_l)$ has no m-step prey whereas the set m-step preys of the vertex $\phi(v_i)$ is $\{\psi_m(C_l) \mid v_i \in C_l\}$. Thus $C^m(D) = G \cup I_{m\theta}$. We take $d = n$ and $k = m\theta$ to complete the proof. □

By Proposition 3.4 and Theorem 3.5, we can define the notion the partial order m-step competition dimension of a graph.

DEFINITION 3.6. For a graph G and a positive integer m, the *partial order m-step competition dimension* $\dim_{\text{poc}}(G; m)$ of G is defined as the smallest positive integer d such that G together with k additional isolated vertices is the m-step competition graph of some d-partial order and some nonnegative integer k, i.e.,

$$\dim_{\text{poc}}(G; m) = \min\{d \in \mathbb{Z}_{>0} \mid \exists k \in \mathbb{Z}_{\geq 0}, \exists S \subset \mathbb{R}^d, \text{ s.t. } G \cup I_k = C^m(D_S)\}.$$

For every graph G, it easily follows from the definition that $\dim_{\text{poc}}(G; 1) = \dim_{\text{poc}}(G)$.

Proposition 3.7. *For a graph G and a positive integer m,* $\dim_{\text{poc}}(G; m) \leq |V(G)|$.

Proof. It follows from the construction of D_V in the proof of Theorem 3.5. □

Choi *et al.* [6] characterized the graphs having partial order competition dimensions 1 or 2, and then presented some graphs having partial order competition dimensions at most three.

Proposition 3.8 ([6]). *For a graph G,* $\dim_{\text{poc}}(G) = 1$ *if and only if $G = K_t$ or $G = K_t \cup K_1$ for some positive integer t.*

Proposition 3.9 ([6]). *For a graph G,* $\dim_{\text{poc}}(G) = 2$ *if and only if G is an interval graph which is neither K_t nor $K_t \cup K_1$ for any positive integer t.*

It is natural to ask which graphs have small partial order m-step competition dimensions. It is easy to characterize graphs G with $\dim_{\text{poc}}(G; m) \leq 1$ for a given positive integer m.

Proposition 3.10. *Let G be a graph and m be a positive integer m. Then* $\dim_{\text{poc}}(G; m) = 1$ *if and only if $G = K_t \cup I_s$ for some nonnegative integers t and s with $t \geq 1$ and $s \leq m$.*

Proof. ($\Rightarrow$) Assume $\dim_{\text{poc}}(G; m) = 1$. Then $G \cup I_k = C^m(D)$ for some 1-partial order D and $k \in \mathbb{Z}_{\geq 0}$. Let $v_1, \ldots, v_n$ be the vertices of D. We may assume that $v_1 < v_2 < \cdots < v_n$ in $\mathbb{R}$. Then, the vertices $v_1, v_2, \ldots, v_m$ does not have an m-step prey in D, so they are isolated in $C^m(D)$. In addition, the vertices $v_{m+1}, v_{m+2}, \ldots, v_n$ has v_1 as an m-step prey, so they form a clique in $C^m(D)$. Therefore, $C^m(D)$ consists of a clique together with some isolated vertices.

($\Leftarrow$) Assume $G = K_t \cup I_s$ for some $t \geq 1$ and $s \leq m$. We denote the vertices in K_t by $x_1, \ldots, x_t$ and the vertices in I_s by $y_1, \ldots, y_s$ if $s \neq 0$. We assign a coordinate in $\mathbb{R}$ to each vertex of G by $y_i = i$ for $i = 1, \ldots, s$ and $x_j = j + s$ for $j = 1, \ldots, t$. Let J be a set of $m - s$ negative real numbers. Then the set $V(G) \cup J \subset \mathbb{R}$ induces a 1-partial order whose m-step competition graph is $G \cup I_{m-s}$. Therefore $\dim_{\text{poc}}(G; m) \leq 1$. □

Park *et al.* [11] studied the m-step competition graphs of 2-partial orders and obtained the following results.

THEOREM 3.11 ([11]). *For a positive integer m, the m-step competition graph of a 2-partial order is an interval graph.*

THEOREM 3.12 ([11]). *For a positive integer m, an interval graph together with some additional vertices is the m-step competition graph of a 2-partial order.*

We can restated the results of Park *et al.* [11] in terms of $\dim_{\mathrm{poc}}(G; m)$ as follows:

PROPOSITION 3.13. *For a graph G and a positive integer m, $\dim_{\mathrm{poc}}(G; m) \leq 2$ if and only if G is an interval graph.*

Proof. ($\Rightarrow$) Assume $\dim_{\mathrm{poc}}(G; m) \leq 2$. Then $G \cup I_k = C^m(D)$ for some 2-partial order D and $k \in \mathbb{Z}_{\geq 0}$. By Theorem 3.11, $G \cup I_k$ is an interval graph and so is G.
($\Leftarrow$) It immediately follows from Theorem 3.12. □

The following proposition tells us that deleting isolated vertices from a graph does not increase the partial order m-step competition dimension.

PROPOSITION 3.14. *For a graph G and positive integers k and m, $\dim_{\mathrm{poc}}(G; m) \leq \dim_{\mathrm{poc}}(G \cup I_k; m)$.*

Proof. Let $d = \dim_{\mathrm{poc}}(G \cup I_k; m)$. Then $(G \cup I_k) \cup I_s = C^m(D)$ for some d-partial order D and $s \in \mathbb{Z}_{\geq 0}$. Since $(G \cup I_k) \cup I_s = G \cup I_{k+s}$, $\dim_{\mathrm{poc}}(G; m) \leq d$. □

As a matter of fact, the equality in Proposition 3.14 mostly holds except for some specific graphs.

PROPOSITION 3.15. *For a graph G and positive integers m and k, $\dim_{\mathrm{poc}}(G \cup I_k; m) > \dim_{\mathrm{poc}}(G; m)$ if and only if $G = K_t \cup I_s$ for some nonnegative integers t and s with $t \geq 1$ and $m - k < s \leq m$.*

Proof. ($\Leftarrow$) Suppose $G = K_t \cup I_s$ for some nonnegative integers t and s with $t \geq 1$ and $m - k < s \leq m$. Since $G \cup I_k = K_t \cup I_{s+k}$ and $s + k > m \geq s$, Proposition 3.10 tells us that $\dim_{\mathrm{poc}}(G; m) = 1 < \dim_{\mathrm{poc}}(G \cup I_k; m)$.
($\Rightarrow$) Let $d = \dim_{\mathrm{poc}}(G; m)$. Then $G \cup I_s = C^m(D)$ for some d-partial order D and $s \in \mathbb{Z}_{\geq 0}$. Suppose, to the contrary, that $d \geq 2$. Let

$$\alpha = \max\{v_1 \mid (v_1, v_2, \ldots, v_d) \in V(D)\},$$
$$\beta = \min\{v_2 \mid (v_1, v_2, \ldots, v_d) \in V(D)\}.$$

Let $z_i = (\alpha + i, \beta - i, 0, \ldots, 0) \in \mathbb{R}^d$ for each $i = 1, \ldots, k$ and let $S = V(D) \cup \{z_1, \ldots, z_k\}$. Then D_S is a d-partial order. By definition, no vertex in $\{z_1, \ldots, z_k\}$ is comparable with any vertex of D_S in $\mathbb{R}^d$. Therefore $C^m(D_S) = C^m(D) \cup I_k = (G \cup I_s) \cup I_k = (G \cup I_k) \cup I_s$. Thus $\dim_{\text{poc}}(G \cup I_k; m) \leq d$, which contradicts the hypothesis that $\dim_{\text{poc}}(G \cup I_k; m) > \dim_{\text{poc}}(G; m)$. Hence $d = 1$. By Proposition 3.10, $G = K_t \cup I_s$ for some nonnegative integers t and s with $t \geq 1$ and $s \leq m$. Then $G \cup I_k = (K_t \cup I_s) \cup I_k = K_t \cup I_{s+k}$. If $s + k \leq m$, then $\dim_{\text{poc}}(G \cup I_k; m) = 1$ by Proposition 3.10 and this contradicts the assumption that $\dim_{\text{poc}}(G \cup I_k; m) > \dim_{\text{poc}}(G; m) = 1$. Therefore $s + k > m$ or $m - k < s$. □

4. $\dim_{\text{poc}}(G; m)$ in the aspect of $\dim_{\text{poc}}(G)$

In this section, we will investigate the behavior of $\dim_{\text{poc}}(G; m)$ when m varies and then present a relation between $\dim_{\text{poc}}(G; m)$ and $\dim_{\text{poc}}(G)$.

Definition 4.1. A d-partial order D is said to satisfy the *distinct coordinate property* (*DC-property* for short) provided that, for each $i = 1, \ldots, d$, the ith coordinates of the vertices of D are all distinct.

For example, the 3-partial order on the three vertices $(1, 2, 3)$, $(2, 3, 4)$, $(3, 4, 5)$ satisfies the DC-property while the 3-partial order on the three vertices $(1, 2, 3)$, $(2, 3, 4)$, $(1, 4, 5)$ does not satisfies the DC-property.

For a d-partial order D and an ordered pair $(i, k) \in \{1, \ldots, d\} \times \mathbb{R}$, we partition $V(D)$ into three disjoint subsets

$$V_{i,k}(D) = \{(a_1, \ldots, a_d) \in V(D) \mid a_i = k\},$$
$$V^+_{i,k}(D) = \{(a_1, \ldots, a_d) \in V(D) \mid a_i > k\},$$
$$V^-_{i,k}(D) = \{(a_1, \ldots, a_d) \in V(D) \mid a_i < k\},$$

and let $\Gamma(D) = \{(i, k) \in \{1, \ldots, d\} \times \mathbb{R} \mid |V_{i,k}(D)| \geq 2\}$. Clearly, a d-partial order D satisfies the DC-property if and only if $\Gamma(D) = \emptyset$.

Proposition 4.2. *For a positive integer d, every d-partial order is isomorphic to a d-partial order satisfying the DC-property.*

Proof. Let D be a d-partial order. There is nothing to prove if $\Gamma(D) = \emptyset$. Assume $\Gamma(D) \neq \emptyset$. Take $(i, k) \in \Gamma(D)$. Let $V_{i,k} = \{v_1, \ldots, v_l\}$ $(l \geq 2)$ and $V^*_{i,k} = \{v^*_1, \ldots, v^*_\ell\}$ where v^*_j is the point in $\mathbb{R}^{d-1}$ obtained from $v_j \in \mathbb{R}^d$ by deleting its ith coordinate. Then $V^*_{i,k}$ induces a $(d-1)$-partial order D^*, i.e., $D^* = D_{V^*_{i,k}}$. Since D^* is acyclic, we may assume that the vertices in $V_{i,k}$ are labeled so that $v_j^* \prec v_{j'}^*$ in D^* only if $j < j'$. Now we construct a new d-partial order $D_{i,k}$ with the vertex set $\{\phi_{i,k}(v) \in \mathbb{R}^d \mid v \in V(D)\}$ so that

$$\phi_{i,k}(v) = \begin{cases} v & \text{if } v \in V_{i,k}^-, \\ v + je_i, & \text{if } v \in V_{i,k} \text{ and } v = v_j, \\ v + le_i & \text{if } v \in V_{i,k}^+, \end{cases}$$

where e_i denotes the ith standard basis vector in $\mathbb{R}^d$. By the way of construction, $D_{i,k}$ is isomorphic to D and $|\Gamma(D_{i,k})| = |\Gamma(D)| - 1$. If $\Gamma(D_{i,k}) = \emptyset$, then $D_{i,k}$ is a desired d-partial order. Otherwise, we repeat this process until we obtain a d-partial order D' which is isomorphic to D and satisfies $\Gamma(D') = \emptyset$. □

The *length* of a directed path P is the number of arcs in P, and denoted by $\ell(P)$.

LEMMA 4.3. *Let G be the m-step competition graph of a d-partial D for some positive integers m and d. If two vertices u and v are adjacent in G, then they have an m-step common prey which has outdegree* 0 *in D.*

Proof. Take two adjacent vertices u and v in G. By the definition $C^m(D)$, u and v have an m-step common prey, say z, in D. Take a longest directed path P starting from z in D and let w be its terminus. It is clear that w has outdegree 0 in D and w is an $(m + \ell(P))$-step common prey of u and v. Then w is an m-step common prey of u and v by Lemma 3.2. □

The following theorem is one of our main results.

THEOREM 4.4. *For a graph G and a positive integer m,* $\dim_{\text{poc}}(G; m) \geq \dim_{\text{poc}}(G; m + 1)$.

Proof. Let $d = \dim_{\text{poc}}(G; m)$. Then $G \cup I_k = C^m(D)$ for some d-partial order D and $k \in \mathbb{Z}_{\geq 0}$. By Proposition 4.2, we may assume D satisfies the DC-property. Then

$$\delta := \min_i\{|a_i - b_i| : (a_1, \dots, a_d) \text{ and } (b_1, \dots, b_d) \text{ are distinct vertices of } D\}$$

is a positive real number. Let Y be the set of vertices of D with outdegree 0. Since D is acyclic, $Y \neq \emptyset$. For each $y \in Y$, let $\phi(y) = y - \frac{\delta}{2}(1, \dots, 1) \in \mathbb{R}^d$ and $Z = \{\phi(y) \mid y \in Y\}$. Then the set $S := V(D) \cup Z$ induces the d-partial order D_S. By the transitivity of D and by the choice of δ, it is easy to see that $N_{D_S}^-(\phi(y)) = \{y\} \cup N_D^-(y)$ and $N_{D_S}^+(\phi(y)) = \emptyset$ for each

$y \in Y$. Furthermore, the set of vertices of outdegree 0 in D_S is Z and the set of vertices of outdegree 1 in D_S is Y.

We claim that $C^m(D)$ and $C^{m+1}(D_S)$ have the same edge set. Take an edge uv in $C^m(D)$. By Lemma 4.3, u and v have a common m-step prey y which has outdegree 0 in D. Since $y \in Y$, $y \to \phi(y)$ in D_S and so $\phi(y)$ is an $(m+1)$-step common prey of u and v in D_S. Thus uv is an edge in $C^{m+1}(D_S)$.

Conversely, take an edge uv in $C^{m+1}(D_S)$. By Lemma 4.3, u and v have an $(m+1)$-step common prey z which has outdegree 0 in D_S. Then there exist two directed paths

$$P_u : u = u_0 \to u_1 \to \cdots \to u_{m-1} \to u_m \to u_{m+1} = z$$

and

$$P_v : v = v_0 \to v_1 \to \cdots \to v_{m-1} \to v_m \to v_{m+1} = z$$

of length $m+1$ in D_S. Since Z is the set of vertices of D_S with outdegree 0, $z \in Z$ and so $z = \phi(y)$ for some $y \in Y$. Since D_S is transitive and $u_{m-1} \to u_m \to \phi(y)$ in D_S, we have $u_{m-1} \to \phi(y)$. Then $u_{m-1} \in N^-_{D_S}(\phi(y)) = \{y\} \cup N^-_D(y)$. However, $u_{m-1} \neq y$, for otherwise u_{m-1} has outdegree 1 in D_S, which is impossible as $u_{m-1} \to u_m$ and $u_{m-1} \to \phi(y)$. Therefore $u_{m-1} \in N^-_D(y)$. Thus the sequence $P'_u : u = u_0 \to u_1 \to \cdots \to u_{m-1} \to y$ is a directed path in D of length m. Similarly, $P'_v : v = v_0 \to v_1 \to \cdots \to v_{m-1} \to y$ is a directed path in D of length m. Then y is an m-step common prey of u and v in D, and therefore uv is an edge in $C^m(D)$.

We have shown that $C^m(D)$ and $C^{m+1}(D_S)$ have the same edge set. Since $C^m(D) = G \cup I_k$, we have $C^{m+1}(D_S) = (G \cup I_k) \cup I_\ell = G \cup I_{k+\ell}$ where $\ell = |Z|$. Hence $\dim_{\text{poc}}(G; m+1) \leq d$. □

By applying induction on m, we have the following corollary.

Corollary 4.5. *For every graph G and every positive integer m, $\dim_{\text{poc}}(G) \geq \dim_{\text{poc}}(G; m)$.*

5. Partial order competition exponents of graphs

In this section, we introduce an analogue concept of exponent for a graph in the aspect of partial order m-step competition dimension.

It is well known that, for a $\{0,1\}$-matrix A with Boolean operation, the matrix sequence $\{A^m\}_{m=1}^{\infty}$ converges to the all-one matrix J if and only if A is primitive. The smallest positive integer M satisfying $A^m = J$ for all $m \geq M$ is called the *exponent* of A.

Let G be a graph. Then the integer-valued sequence $\{\dim_{\text{poc}}(G;m)\}_{m=1}^{\infty}$ is bounded by Proposition 3.7 and decreasing by Theorem 4.4. Therefore there exists a positive integer M such that $\dim_{\text{poc}}(G;m)$ is constant for any $m \geq M$. We call the smallest such M the *partial order competition exponent* of G and denote it by $\exp_{\text{poc}}(G)$.

PROPOSITION 5.1. *For any graph G with* $\dim_{\text{poc}}(G;1) = 1$, $\exp_{\text{poc}}(G) = 1$.

Proof. Since $\{\dim_{\text{poc}}(G;m)\}_{m=1}^{\infty}$ is decreasing, $1 = \dim_{\text{poc}}(G;1) \geq \dim_{\text{poc}}(G;2) \geq \cdots$ and so $\dim_{\text{poc}}(G;m) = 1$ for any $m \in \mathbb{Z}_{>0}$. Therefore $\exp_{\text{poc}}(G) = 1$. □

PROPOSITION 5.2. *For any positive integer M, there exists a graph G such that* $\dim_{\text{poc}}(G;1) = 2$ *and* $\exp_{\text{poc}}(G) = M$.

Proof. Let G be an interval graph which is not of the form $K_t \cup I_s$ for any $t \in \mathbb{Z}_{>0}$ and $s \in \mathbb{Z}_{\geq 0}$. Then $\dim_{\text{poc}}(G;m) = 2$ for any $m \in \mathbb{Z}_{>0}$ by Propositions 3.10 and 3.13. Therefore $\exp_{\text{poc}}(G) = 1$.

Take a positive integer $M \geq 2$. Consider $H = K_t \cup I_M$ where t is an arbitrary positive integer. Then, by Proposition 3.10, $\dim_{\text{poc}}(H;M-1) = 2$ and $\dim_{\text{poc}}(H;M) = 1$. Therefore $\exp_{\text{poc}}(H) = M$. □

PROPOSITION 5.3. *For any graph G with* $\dim_{\text{poc}}(G;1) = 3$, $\exp_{\text{poc}}(G) = 1$.

Proof. Since $\dim_{\text{poc}}(G;1) = 3 > 2$, G is not an interval graph as shown by As shown by Cho and Kim [3]. Thus $\dim_{\text{poc}}(G;m) > 2$ for any $m \in \mathbb{Z}_{>0}$ by Proposition 3.13. On the other hand, by Corollary 4.5, $\dim_{\text{poc}}(G;m) \leq \dim_{\text{poc}}(G;1) = 3$ and so $\dim_{\text{poc}}(G;m) = 3$ for any $m \in \mathbb{Z}_{>0}$. Thus $\exp_{\text{poc}}(G) = 1$. □

6. Acknowledgement

The material in this paper is from the author's Ph.D. thesis [5].

References

[1] E. Belmont, *A complete characterization of paths that are m-step competition graphs,* Discrete Applied Mathematics, **159** (2011), no. 14, 1381-1390.

[2] H.H. Cho and H.K. Kim, *Competition indices of strongly connected digraphs,* Bull. Korean Math. Soc., **48** (2011), no. 3, 637-646.

[3] Han Hyuk Cho and S.-R. Kim, *A class of acyclic digraphs with interval competition graphs,* Discrete Applied Mathematics, **148** (2005), no. 2, 171-180.

[4] H.H. Cho, S.-R. Kim, and Y. Nam, *The m-step competition graph of a digraph,* Discrete Applied Mathematics, **105** (2000), no. 1, 115-127.

[5] J. Choi, *A study on the competition graphs of d-partial orders,* PhD thesis, Seoul National University, 2018.

[6] J. Choi, K.S. Kim, S.-R. Kim, J.Y. Lee, and Y. Sano, *On the competition graphs of d-partial orders,* Discrete Applied Mathematics, **204** (2016), 29-37.

[7] J. Choi and S.-R. Kim, *On the matrix sequence for a boolean matrix a whose digraph is linearly connected,* Linear Algebra and its Applications, **450** (2014), 56-75.

[8] G.T Helleloid, *Connected triangle-free m-step competition graphs,* Discrete Applied Mathematics, **145** (2005), no. 3, 376-383.

[9] W. Ho, *The m-step, same-step, and any-step competition graphs,* Discrete Applied Mathematics, **152** (2005), no. 1, 159-175.

[10] H.K. Kim, *Competition indices of tournaments,* Bull. Korean Math. Soc, **45** (2008), no. 2, 385-396.

[11] B. Park, J.Y. Lee, and S.-R. Kim, *The m-step competition graphs of doubly partial orders,* Applied Mathematics Letters, **24** (2011), no. 6, 811-816.

[12] W. Park, B. Park, and S.-R. Kim, *A matrix sequence* $\{\Gamma$ *(Am)*$\}$ *m= 1 might converge even if the matrix a is not primitive,* Linear Algebra and its Applications, **438** (2013), no. 5, 2306-2319.

[13] Y. Zhao and G.J Chang, *Note on the m-step competition numbers of paths and cycles,* Discrete Applied Mathematics, **157** (2009), no. 8, 1953-1958.

*
Department of Mathematics Education
Cheongju University
Cheongju 28503, Republic of Korea
E-mail: jihoon@cju.ac.kr