JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **32**, No. 2, May 2019
http://dx.doi.org/10.14403/jcms.2019.32.2.251

THE ARTINIAN COMPLETE INTERSECTION QUOTIENT AND THE STRONG LEFSCHETZ PROPERTY

Yong-Su Shin*

Abstract. It has been little known when an Artinian (point) quotient has the strong Lefschetz property. In this paper, we find the Artinian complete intersection quotient having the SLP. More precisely, we prove that if $\mathbb{X}$ is a complete intersection in $\mathbb{P}^2$ of type $(2,2)$ and $\mathbb{Y}$ is a finite set of points in $\mathbb{P}^2$ such that $\mathbb{X} \cup \mathbb{Y}$ is a basic configuration of type $(2,a)$ with $a \geq 3$ or $(3,a)$ with $a = 3,4,5,6$, then $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP. We also show that if $\mathbb{X}$ is a complete intersection in $\mathbb{P}^2$ of type $(3,2)$ and $\mathbb{Y}$ is a finite set of points in $\mathbb{P}^2$ such that $\mathbb{X} \cup \mathbb{Y}$ is a basic configuration of type $(3,3)$ or $(3,4)$, then $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP.

1. Introduction

Let $R = \Bbbk[x_1, \dots, x_n]$ be an n-variable polynomial ring over a field $\Bbbk$ of characteristic 0. A standard graded $\Bbbk$-algebra $A = R/I = \oplus_{i \geq 0} A_i$ has the *weak Lefschetz property* (WLP) if there is a linear form ℓ such that the multiplication $\times \ell : A_i \to A_{i+1}$ has maximal rank for every $i \geq 0$, and A has the *strong Lefschetz property* (SLP) if $\times \ell^d : A_i \to A_{i+d}$ has maximal rank for every $i \geq 0$ and $d \geq 1$. In this case, ℓ is called a *strong Lefschetz element* of A. If $d = 1$, then ℓ is a *weak Lefschetz element* of A.

The WLP and SLP are closely connected to several topics in algebraic geometry, commutative algebra, combinatorics, and representation theory ([7, 8, 9, 10, 11, 12, 15]). In [7], Harima et al. gives an overview of the Lefschetz properties from a different prospective focusing on representation theory, and provides a wonderfully comprehensive exploration

Received March 17, 2019; Accepted May 08, 2019.
2010 Mathematics Subject Classification: 13A02, 16W50.
Key words and phrases: Hilbert functions, weak Lefschetz property, strong Lefschetz property, complete intersection.
This paper was supported by a grant from Sungshin Women's University.

of the Lefschetz properties using the Jordan type. Here the Jordan type of $\ell \in \mathfrak{m}$ is the partition giving the Jordan blocks of the multiplication map $\times\ell : M \to M$, where M is a module of A, and $\mathfrak{m}$ is a unique maximal homogeneous ideal of R (see [7, 11]).

Recently in 2018, Iarrobino, Marques, and McDaniel [9] explored a general invariant of an Artinian Gorenstein algebra A or A-module M, which is the set of Jordan types of elements of the maximal ideal $\mathfrak{m}$ of A. In [1], the authors showed that if $\mathbb{X}$ and $\mathbb{Y}$ are linear star configurations in $\mathbb{P}^2$ (see [13, 14] for the definition of a star configuration), then the Artinian quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the WLP. Indeed, this is true in $\mathbb{P}^n$ in general. Moreover, in [11], the authors proved that if $\mathbb{X}$ is a specific $\Bbbk$-configuration in $\mathbb{P}^2$ (see [2] for the definition of a $\Bbbk$-configuration) and $\mathbb{Y}$ is a finite set of points in $\mathbb{P}^2$ such that $\mathbb{X} \cup \mathbb{Y}$ is a basic configuration in $\mathbb{P}^2$, then the Artinian quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP.

In this paper, we focus on the following question.

Question 1.1. Let $\mathbb{X}$ be a complete intersection in $\mathbb{P}^2$ contained in a basic configuration $\mathbb{Z}$ (see Definition 2.2) and let $\mathbb{Y} := \mathbb{Z} - \mathbb{X}$.

(a) Does the Artinian quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ have the WLP?
(b) Does the Artinian quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ have the SLP?

For Question 1.1 (a), "most" Artinian Gorenstein rings have the WLP (see [4, 16]). However, it is not much known if Question 1.1 (b) is true.

In Section 2, we introduce two basic results to show if the Artinian quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP. The first one is a Waring decomposition, and the second one is the Jordan type argument. In Section 3, we prove that if $\mathbb{X}$ is a complete intersection in $\mathbb{P}^2$ of type $(2, 2)$ or $(2, 3)$, then some of the Artinian quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP (see Theorems 3.1, 3.2). In particular, we give a complete answer to Question 1.1 (b) when $\mathbb{X}$ is a complete intersection in $\mathbb{P}^2$ of type $(2, 2)$ and $\mathbb{Z}$ is a basic configuration in $\mathbb{P}^2$ of type $(2, a)$ with $a \geq 3$ (see Proposition 3.3). Our results are far away to the full answer to Question 1.1, and so we might try more to obtain the general answer to this question.

We linked full calculations for this paper to Arxiv to make this paper shortened (see complete intersection quotient-fulltext.pdf).

2. A Waring decomposition and the Jordan type

Remark 2.1 ([11, Remark 2.1]). Let $\Bbbk$ be a field of characteristic zero and let $F \in \Bbbk[x_0, x_1, \ldots, x_n] = R = \bigoplus_{i \geq 0} R_i$ $(n \geq 1)$ be a homogeneous polynomial (form) of degree d, i.e., $F \in R_d$. It is well known that in this case each R_i has a basis consisting of i-th powers of linear forms. Thus we may write

$$F = \sum_{i=1}^{r} \alpha_i L_i^d, \qquad \alpha_i \in \Bbbk,\ L_i \in R_1.$$

If $\Bbbk$ is algebraically closed (which we now assume for the rest of the paper) then each $\alpha_i = \beta_i^d$ for some $\beta_i \in \Bbbk$ and so we can write

$$F = \sum_{i=1}^{r} (\beta_i L_i)^d = \sum_{i=1}^{r} M_i^d, \quad M_i \in R_1. \tag{2.1}$$

We call a description of F as in equation (2.1), a *Waring decomposition* of F. The least integer r such that F has a Waring decomposition with exactly r summands is called the *Waring rank* (or simply the *rank*) of F.

Definition 2.2 ([6, Definition 3.1]). A finite set $\mathbb{X}$ of points in $\mathbb{P}^2$ is called a *basic configuration* of type (d, e) if there exist distinct element b_j, c_j in $\Bbbk$ such that

$$I_{\mathbb{X}} = \left(\prod_{j=1}^{d} (x - b_j z), \prod_{j=1}^{e} (x - c_j z) \right).$$

We now introduce two important tools to verify if an Artinain ring has the WLP or SLP.

Lemma 2.3 ([7]). *Assume A is graded and $\mathbf{H}_A$ is unimodal. Then*

(a) *A has the WLP if and only if the number of parts of $J_{\ell,A} = \max\{\mathbf{H}_A(i)\}$. (The Sperner number of A);*
(b) *ℓ is a strong Lefschetz element of A if and only if $J_{\ell,A} = \mathbf{H}_A^\vee$ the conjugate of $\mathbf{H}_A$ (exchange rows and columns in the Ferrers diagram of $\mathbf{H}_A$).*

Proposition 2.4 ([3, Proposition 5.15]). *Let $\mathbb{X}$ be a finite set of points in $\mathbb{P}^n$ and let A be an Artinian quotient of the coordinate ring of $\mathbb{X}$. Assume that $\mathbf{H}_A(i) = \mathbf{H}_{\mathbb{X}}(i)$ for all $0 \le i \le \sigma(\mathbb{X}) - 1$. Then A has the WLP.*

3. The strong Lefschetz property

In this section, we shall introduce the Artinian complete intersection quotient having the SLP.

Let $R = \Bbbk[x, y, z]$. First, note that $\mathbb{L}_i$ is the line defined by a linear form $L_i = x - (i-1)z$ and $\mathbb{M}_j$ is the line defined by a linear form $M_j = y - (j-1)z$ for i and j, and that $\wp_{i,j}$ is a point defined by two linear forms L_i and M_j.

Theorem 3.1. *Let $\mathbb{X}$ be a complete intersection in $\mathbb{P}^2$ of type $(2, 2)$ contained in a basic configuration $\mathbb{Z}$ in $\mathbb{P}^2$ of type $(2, a)$ with $a \geq 3$. Define $\mathbb{Y} := \mathbb{Z} - \mathbb{X}$, ($\mathbb{X}$ is a set of solid 4-points in Figure 1).*

$$\begin{array}{cccccc} \bullet & \bullet & \circ & \cdots & \circ & \mathbb{L}_2 \\ \bullet & \bullet & \circ & \cdots & \circ & \mathbb{L}_1 \\ \mathbb{M}_1 & \mathbb{M}_2 & \mathbb{M}_3 & \cdots & \mathbb{M}_a & \end{array}$$

Figure 1.

Then the Artinian complete intersection quotient $A := R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP.

Proof. Since all cases can be proved by an analogous argument, we shall introduce a proof for $a \geq 5$ only here. We linked the full calculations for this proof to Arxiv (see complete intersection quotient-fulltext.pdf).

Suppose $a \geq 5$. Then by [5, Theorem 2.1] the Hilbert function of A is

$$\mathbf{H}_A \quad : \quad 1 \quad 3 \quad 4 \quad \cdots \quad \overset{(a-3)\text{-rd}}{4} \quad 3 \quad 1 \quad 0.$$

By [6, Lemma 3.4],

$$I_{\mathbb{X}} + I_{\mathbb{Y}} = (L_1L_2, M_1M_2, M_3 \cdots M_a).$$

By a Waring decomposition, for a general linear form ℓ,

$$\ell^{a-1} \notin [I_{\mathbb{X}} + I_{\mathbb{Y}}]_{a-1}.$$

Thus the Jordan type $J_{\ell,A}$ is of the form

$$J_{\ell,A} = (a, \dots).$$

Without loss of generality, we may assume that a linear form ℓ vanishes on a point $\wp_{1,1}$, but ℓ does not vanish on any other points $\wp_{i.j}$ for $(i, j) \neq (1, 1)$. Assume that

$$(\alpha L_1 + \beta M_2 + \gamma \ell)\ell^{a-3} = F_1L_1L_2 + F_2M_1M_2 + \delta M_3 \cdots M_a \in [I_{\mathbb{X}} + I_{\mathbb{Y}}]_{a-2},$$

for some $F_i \in R_{a-4}$, and $\delta \in \Bbbk$. Since a form $F_1L_1L_2 + F_2M_1M_2 + \delta M_3 \cdots M_a$ has to vanish on the point $\wp_{1,1}$, we see that $\delta = 0$, and so

$$(\alpha L_1 + \beta M_2 + \gamma \ell)\ell^{a-3} = F_1L_1L_2 + F_2M_1M_2.$$

By the analogous argument as before, one can easily find that $\alpha = \beta = \gamma = 0$, i.e., the 3-forms

$$L_1\ell^{a-3}, M_2\ell^{a-3}, \ell^{a-2}$$

are linearly independent. It follows that for a general linear form ℓ the Jordan type $J_{\ell,A}$ is of the form

$$J_{\ell,A} = (a, a-2, a-2, \dots).$$

Notice that by [3, Proposition 5.15] the Artinian ring A has the WLP, and so by Lemma 2.3 (a) the number of partitions of the Jordan type $J_{\ell,A}$ is the same as the Sperner number. Thus

$$J_{\ell,A} = \mathbf{H}_A^\vee = (a, a-2, a-2, a-4),$$

as we desired. Therefore, by Lemma 2.3 (b), the Artinian ring A has the SLP. □

THEOREM 3.2. *Let $\mathbb{X}$ be a complete intersection in $\mathbb{P}^2$ of type $(2,2)$ contained in a basic configuration $\mathbb{Z}$ in $\mathbb{P}^2$ of type $(3,a)$ with $a = 3,4,5,6$. Define $\mathbb{Y} := \mathbb{Z} - \mathbb{X}$, ($\mathbb{X}$ is a set of solid 4-points in Figure 2).*

$$\begin{array}{cccccc}
\circ & \circ & \circ & \cdots & \circ & \mathbb{L}_3 \\
\bullet & \bullet & \circ & \cdots & \circ & \mathbb{L}_2 \\
\bullet & \bullet & \circ & \cdots & \circ & \mathbb{L}_1 \\
\mathbb{M}_1 & \mathbb{M}_2 & \mathbb{M}_3 & \cdots & \mathbb{M}_a &
\end{array}$$

FIGURE 2.

Then the Artinian complete intersection quotient $A := R/(I_\mathbb{X} + I_\mathbb{Y})$ has the SLP.

Proof. Since all cases can be proved by a similar argument, we shall introduce a brief proof for $a = 6$ only here. We linked the full calculations for this proof to Arxiv (see complete intersection quotient-fulltext.pdf).

Consider $a = 6$. Note that by [5, Theorem 2.1] the Hilbert function of A is

$$\mathbf{H}_A \ : \ 1 \quad 3 \quad 4 \quad 4 \quad 4 \quad 3 \quad 1 \quad 0.$$

By [6, Lemma 3.4],

$$\begin{array}{lcl}
I_\mathbb{X} & = & (L_1L_2, M_1M_2), \qquad \text{and} \\
I_\mathbb{Y} & = & (L_1L_2L_3, M_1M_2\cdots M_6, L_3M_3M_4M_5M_6).
\end{array}$$

Hence the ideal $I_\mathbb{X} + I_\mathbb{Y}$ has 3-minimal generators, i.e.,

$$I_\mathbb{X} + I_\mathbb{Y} \ = \ (L_1L_2, M_1M_2, L_3M_3M_4M_5M_6).$$

By a Waring decomposition, for a general linear form ℓ, the Jordan type $J_{\ell,A}$ is of the form

$$J_{\ell,A} = (7, \dots).$$

Without loss of generality, we may assume that a linear form $\ell := z$. Now assume that

(3.1)

$$(\alpha x + \beta y + \gamma z)z^4 = F_1 L_1 L_2 + F_2 M_1 M_2 + \delta L_3 M_3 M_4 M_5 M_6 \in [I_{\mathbb{X}} + I_{\mathbb{Y}}]_5$$

for some $\alpha, \beta, \gamma, \delta \in \Bbbk$ and $F_1, F_2 \in R_3$. Let

$$F_1 = \sum_{i+j+k=3} a_{i,j,k} x^i y^j z^k \quad \text{and} \quad F_2 = \sum_{i+j+k=3} b_{i,j,k} x^i y^j z^k.$$

Since a form $(\alpha x + \beta y + \gamma z)z^4$ vanishes on any point $(x, y, 0)$, we can rewrite equation (3.1) as

$$\sum_{i+j=3} a_{i,j,0} x^{i+2} y^j + \sum_{i+j=3} b_{i,j,0} x^i y^{j+2} + \delta x y^4 = 0.$$

This implies that

$$\begin{array}{rcl} a_{3,0,0} = a_{2,1,0} = b_{0,3,0} & = & 0, \\ a_{1,2,0} + b_{3,0,0} & = & 0, \\ a_{0,3,0} + b_{2,1,0} & = & 0, \quad \text{and} \\ b_{1,2,0} + \delta & = & 0. \end{array}$$

For our convenience, set

$$\begin{array}{l} a_{2,0,1} = a, \quad a_{1,2,0} = b, \quad a_{1,1,1} = c, \quad a_{1,0,2} = d, \\ a_{0,3,0} = e, \quad a_{0,2,1} = f, \quad a_{0,1,2} = g, \quad a_{0,0,3} = h, \\ b_{2,0,1} = i, \quad b_{1,2,0} = j, \quad b_{1,1,1} = k, \quad b_{1,0,2} = l, \\ b_{0,2,1} = m, \quad b_{0,1,2} = n, \quad b_{0,0,3} = p, \quad \text{i.e.,} \end{array}$$

$$\begin{array}{rcl} F_1 & = & ax^2z + bxy^2 + cxyz + dxz^2 + ey^3 + fy^2z + gyz^2 + hz^3, \\ F_2 & = & -bx^3 - ex^2y + ix^2z + jxy^2 + kxyz + lxz^2 + my^2z + nyz^2 + pz^3. \end{array}$$

Thus

$$\begin{array}{rcl} (\alpha x + \beta y + \gamma z)z^4 & = & ax^4z + (b + c)x^3yz + (-b + e + f + i)x^2y^2z \\ & & +(-e + 13j + k)xy^3z + (2j + m)y^4z \\ & & +(-a + d)x^3y^2 + (-c + g - i)x^2yz^2 \\ & & +(-f - 71j - k + l)xy^2z^2 + (-28j - m + n)y^3z^2 \\ & & +(-d + h)x^2z^3 + (-g + 154j - l)xyz^3 \\ & & +(142j - n + p)y^2z^3 - hxz^4 - 120jxz^4 - 308jyz^4 \\ & & -pyz^4 + 240jz^5. \end{array}$$

This indicates that $a = d = h = 0$, and so we have

$$\begin{array}{rcl}(\alpha x + \beta y + \gamma z)z^4 & = & (b+c)x^3yz + (-b+e+f+i)x^2y^2z \\ & & +(-e+13j+k)xy^3z + (2j+m)y^4z \\ & & +(-c+g-i)x^2yz^2 + (-f-71j-k+l)xy^2z^2 \\ & & +(-28j-m+n)y^3z^2 + (-g+154j-l)xyz^3 \\ & & +(142j-n+p)y^2z^3 - 120jxz^4 \\ & & +(-308j-p)yz^4 + 240jz^5.\end{array}$$

We obtain the homogeneous linear system,

$$\begin{array}{rclrclrcl} b+c & = & 0, & -b+e+f+i & = & 0, & -e+13j+k & = & 0, \\ 2j+m & = & 0, & -c+g-i & = & 0, & -f-71j-k+l & = & 0, \\ -28j-m+n & = & 0, & -g+154j-l & = & 0, & 142j-n+p & = & 0. \end{array}$$

Thus we have that $j = p = 0$ (see complete intersection quotient-fulltext.pdf for the calculation with details), and so

$$\begin{array}{rclcl} \alpha & = & -120j & = & 0, \\ \beta & = & -308j-p & = & 0, \quad \text{and} \\ \gamma & = & 240j & = & 0. \end{array}$$

This implies that the 3-forms $\{xz^4, yz^4, z^5\}$ are linearly independent, and thus, for a general linear form ℓ, the 3-forms $\{x\ell^4, y\ell^4, z\ell^4\}$ are linearly independent. Moreover, by Proposition 2.4, A has the WLP, and hence by Lemma 2.3 (a) the Jordan type $J_{\ell,A}$ is

$$J_{\ell,A} = (7,5,5,3) = \mathbf{H}_A^\vee.$$

Theorefore, by Lemma 2.3 (b), the Artinian ring A has the SLP. This completes the proof. □

Proposition 3.3. *Let $R = \Bbbk[x,y,z]$. Let $\mathbb{X}$ be a complete intersection in $\mathbb{P}^2$ of type $(3,2)$ contained in a basic configuration $\mathbb{Z}$ in $\mathbb{P}^2$ of type $(3,a)$ with $a = 3,4$. Define $\mathbb{Y} := \mathbb{Z} - \mathbb{X}$, ($\mathbb{X}$ is a set of solid 6-points in Figure 3).*

$$\begin{array}{cccc} \bullet & \bullet & \circ & \mathbb{L}_3 \\ \bullet & \bullet & \circ & \mathbb{L}_2 \\ \bullet & \bullet & \circ & \mathbb{L}_1 \\ \mathbb{M}_1 & \mathbb{M}_2 & \mathbb{M}_3 & \end{array} \qquad \begin{array}{ccccc} \bullet & \bullet & \circ & \circ & \mathbb{L}_3 \\ \bullet & \bullet & \circ & \circ & \mathbb{L}_2 \\ \bullet & \bullet & \circ & \circ & \mathbb{L}_1 \\ \mathbb{M}_1 & \mathbb{M}_2 & \mathbb{M}_3 & \mathbb{M}_4 & \end{array}$$

Figure 3.

Then the Artinian complete intersection quotient $A := R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP.

Proof. We shall show for $a = 4$ only here (see complete intersection quotient-fulltext.pdf for the case $a = 3$).

Let $a = 4$. By [5, Theorem 2.1] the Hilbert function of $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is

$$\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})} \quad : \quad 1 \quad 3 \quad 4 \quad 3 \quad 1 \quad 0.$$

By a Waring decomposition, for a general linear form ℓ, we have

$$\ell^4 \notin [I_{\mathbb{X}} + I_{\mathbb{Y}}]_4.$$

So the Jordan type $J_{\ell,A}$ is of the form

$$J_{\ell,A} = (5, \dots).$$

Note that

$$I_{\mathbb{X}} + I_{\mathbb{Y}} = (L_1L_3L_3, M_1M_2, M_3M_4).$$

Without loss of generality, we may assume that a linear form $\ell := L_1$. Now assume that

$$(\alpha x + \beta y + \gamma z)L_1^2 = \alpha_1 L_1L_2L_3 + F_1M_1M_2 + F_2M_3M_4 \in [I_{\mathbb{X}} + I_{\mathbb{Y}}]_3$$

for some $\alpha, \beta, \gamma, \alpha_1 \in \Bbbk$, and $F_i \in R_1$. Since a form $(\alpha x + \beta y + \gamma z)L_1^2$ vanishes on two points $\wp_{1,1}$ and $\wp_{1,2}$, the form $F_2M_3M_4$ vanishes on the two points. Hence, by *Bezóut*'s theorem,

$$F_2 = \alpha_3 L_1, \quad \text{for some}, \quad \alpha_3 \in \Bbbk.$$

Thus

$$(\alpha x + \beta y + \gamma z)L_1^2 = \alpha_1 L_1L_2L_3 + F_1M_1M_2 + \alpha_3 L_1M_3M_4.$$

Since $F_1 = \alpha_2 L_1$ for some $\alpha_2 \in \Bbbk$, we get that

$$(\alpha x + \beta y + \gamma z)L_1 = \alpha_1 L_2L_3 + \alpha_2 M_1M_2 + \alpha_3 M_3M_4.$$

Recall that

$$\begin{array}{llll} L_1 = x, & L_2 = x - z, & L_3 = x - 2z, & \\ M_1 = y, & M_2 = y - z, & M_3 = y - 2z, & M_4 = y - 3z. \end{array}$$

Since $L_1 = x$ vanishes on the 3-points $(0, 0, 1), (0, 1, 0)$, and $(0, 1, 1)$, one can easily calculate that

$$\alpha_1 = \alpha_2 = \alpha_3 = 0,$$

and thus

$$\alpha = \beta = \gamma = 0$$

as well. This implies that the 3-forms xL_1^2, yL_1^2, and zL_1^2 are linearly independent, and thus, for a general linear form ℓ, the Jordan type $J_{\ell,A}$ is of the form

$$J_{\ell,A} = (5,3,3,1) = \mathbf{H}_A^{\vee},$$

as we wished. Therefore, by Lemma 2.3, the Artinian ring A has the SLP. □

References

[1] J. Ahn and Y. S. Shin. *The Minimal Free Resolution of A Star-Configuration in $\mathbb{P}^n$ and The Weak-Lefschetz Property*, J. of Korean Math. Soc. **49** (2012), no.2, 405–417.

[2] A. V. Geramita, T. Harima, and Y. S. Shin. *Some Special Configurations of Points in $\mathbb{P}^n$*, J. Algebra, **268** (2003), no. 2, 484–518.

[3] A. V. Geramita, T. Harima, J. C. Migliore, and Y. S. Shin, *The Hilbert function of a level algebra*, Mem. Amer. Math. Soc. **186** (2007), no. 872, vi+139 pp.

[4] A. V. Geramita, T. Harima, and Y. S. Shin. *Extremal point sets and Gorenstein ideals*, Adv. Math. **152** (2000), no. 1, 78–119.

[5] T. Harima, *Some examples of unimodal Gorenstein sequences*, Journal of Pure and Applied Algebra, **103** (1995) 313-324.

[6] T. Harima, *A note on Artinian Gorenstein algebras of codimension three*, J. Pure Appl. Algebra, **135** (1999) 45-56.

[7] T. Harima, T. Maeno, H. Morita, Y. Numata, A. Wachi, and J. Watanabe, *The Lefschetz properties*, Lecture Notes in Mathematics, 2080. Springer, Heidelberg, 2013.

[8] T. Harima, J. Migliore, U. Nagel, and J. Watanabe, *The Weak and Strong Lefschetz Properties for Artinian K-Algebras*, J. Algebra, **262** (2003), 99–126.

[9] A. Iarrobino, P. M. Marques, and C. MaDaniel, *Jordan type and the Associated graded algebra of an Artinian Gorenstein algebra*, arXiv:1802.07383 (2018).

[10] Y. R. Kim and Y. S. Shin, *Star-configurations in $\mathbb{P}^n$ and The Weak-Lefschetz Property*, Comm. Algebra, **44** (2016), no. 9, 3853-3873.

[11] Y. R. Kim and Y. S. Shin, *An Artinian point-configuration quotient and the strong Lefschetz property*, J. Korean Math. Soc. **55** (2018), no. 4, 763-783.

[12] J. Migliore, R. Miró-Roig, *Ideals of general forms and the ubiquity of the Weak Lefschetz Property*, J. Pure Appl. Algebra, **182** (2003), 79–107.

[13] J. P. Park and Y. S. Shin, *The Minimal Free Graded Resolution of A Star-configuration in $\mathbb{P}^n$*, J. Pure Appl. Algebra, **219** (2015), 2124–2133.

[14] Y. S. Shin, *Secants to The Variety of Completely Reducible Forms and The Union of Star-Configurations*, J. Algebra Appl. **11** (2012), 27 pages.

[15] Y. S. Shin, *Star-Configurations in $\mathbb{P}^2$ Having Generic Hilbert Functions and The Weak-Lefschetz Property*, Comm. in Algebra, **40** (2012), 2226-2242.

[16] J. Watanabe, *The Dilworth number of Artinian rings and finite posets with rank function*, Adv. Stud. Pure Math. **11** (1987), 303-312.

*
Department of Mathematics,
Sungshin Women's University,
Seoul, 02844, Republic of Korea
E-mail: ysshin@sungshin.ac.kr