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# MAPPINGS RELATED TO MINIMAL SURFACES

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Abstract. In this paper, we study harmonic mappings related to the nonparametric minimal surfaces that lie over the upper halfplane.

## 1. Introduction

Let $\mathbb{D}$ be a domain in $\mathbb{C}$. A real-valued function $u$ on $\mathbb{D}$ is said to be harmonic in a given domain $\mathbb{D}$ if it has continuous partial derivatives of the first and second order in $\mathbb{D}$ and satisfies the partial differential equation

$$u_{xx} + u_{yy} = 0$$

on $\mathbb{D}$.

A continuous function $f = u + iv$ defined in $\mathbb{D}$ is harmonic if $u$ and $v$ are real harmonic in $\mathbb{D}$. In any simply connected subdomain of $\mathbb{D}$ we can write $f = h + \overline{g}$, where $h$ and $g$ are analytic and $\overline{g}$ denotes the function $z \longmapsto \overline{g(z)}$. A harmonic mapping $f$ is univalent in $\mathbb{D}$ if it is one-to-one and orientation preserving in $\mathbb{D}$.

Let $\Omega$ be a simply connected domain in $\mathbb{C}$. Let $S$ be a nonparametric surface over $\Omega$ given by

$$S = \{(u, v, F(u, v)) : u + iv \in \Omega\}.$$

Then $S$ is a minimal surface if and only if $S$ has the representation of the form

$$S = \left\{ \left( Re \int_0^{\zeta} \phi_1(z)dz + c_1, Re \int_0^{\zeta} \phi_2(z)dz + c_2, Re \int_0^{\zeta} \phi_3(z)dz + c_3 \right) : z \in D \right\}$$

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where

$$\begin{aligned}
(1.1) \qquad & D = \{z : |z| < 1\}, \\
& \phi_1^2 + \phi_2^2 + \phi_3^2 = 0, \\
& \phi_1, \phi_2, \phi_3 \text{ are analytic, and} \\
& f = u + iv = Re \int_0^\zeta \phi_1(z)dz + iRe \int_0^\zeta \phi_2(z)dz + c
\end{aligned}$$

is a conformal univalent harmonic mapping from $D$ onto $\Omega$ [3,4]. Since the mapping $f$ is harmonic in $D$, it is of the form $f = h + \overline{g}$ where $h$ and $g$ are analytic in $D$. In addition, $a = g'/h'$ is analytic in $D$ and $|a(z)| < 1$.

In this paper, we will show that the conformal univalent harmonic mappings

$$\begin{aligned}
f(z) = p_1 + \frac{ip_2}{2}\Bigg[&\left(\frac{1}{2}log\frac{1+z}{1-z} + \frac{z}{(1-z)^2}\right) \\
&- \overline{\left(\frac{1}{2}log\frac{1+z}{1-z} + \frac{z}{(1-z)^2}\right)} + \frac{1+z}{1-z} + \overline{\left(\frac{1+z}{1-z}\right)}\Bigg]
\end{aligned}$$

from $D$ onto the upper halfplane $\Omega = \{w : Im\{w\} > 0\}$ arise in connection with the nonparametric minimal surfaces $S$ that lies over $\Omega$ by using the properties of univalent harmonic mappings.

## 2. Univalent harmonic mapping

The following result is obtained by J.G. Clunie and T. Sheil-Small. We are going to use it in this section.

**Theorem 1**([2]. THEOREM 5.3). *A harmonic $f = h+\overline{g}$ locally univalent in $D$ is a univalent mapping of $D$ onto a domain convex in the direction of the real axis (i.e. a domain which has a connected intersection with every line parallel to the real axis) if and only if $h - g$ is a conformal univalent mapping of $D$ onto a domain convex in the direction of the real axis.*

Let $S$ be a nonparametric minimal surface over $\Omega = \{w : Im\{w\} > 0\}$. Then we have a conformal univalent harmonic mapping $f = h + \overline{g}$ from $D$ onto $\Omega$ satisfying (1.1). Fix a point $p = p_1 + ip_2$ in $\Omega$, and let $P =$

$(p_1, p_2, F(p))$ be the corresponding point of $S$. Since the composition $f \circ \Psi$ of a harmonic function $f$ with an analytic function $\Psi$ is harmonic, we may normalize in such a way that the harmonic mapping $f$ satisfies $f(0) = p$. Since $\Omega$ is convex in the direction of the real axis, we know that the analytic function $\phi = h - g$ is a conformal univalent mapping of $D$ onto the domain $\Omega$ by using Theorem 1. We are free to normalize $g(0) = 0$, in which case $\phi(0) = h(0) = f(0) = p$.

**Theorem 2.** *A conformal univalent harmonic mapping $f = h + \overline{g}$ from $D$ onto $\Omega$ satisfying* (1.1) *normalized by $f(0) = p$ and $g(0) = 0$ has the representation*

$$f(z) = Re\{p + \int_0^z \frac{1+a}{1-a}\phi' dz\} + iIm\{\phi\}. \tag{2.1}$$

**Proof.** $f = h + \overline{g}$ and $a = g'/h'$ implies that $Im\{f\} = Im\{\phi\}$, $Re\{f\} = Re\{h + g\}$, and $h' + g' = \frac{1+a}{1-a}\phi'$.

$$\begin{aligned} f(z) =& Re\{f\} + iIm\{f\} = Re\{h + g\} + iIm\{\phi\} \\ =& Re\{\int_0^z (h' + g')dz + c\} + iIm\{\phi\} \\ =& Re\{\int_0^z \frac{1+a}{1-a}\phi' dz + c\} + iIm\{\phi\}. \end{aligned}$$

From $f(0) = p$ and $g(0) = 0$, we obtain

$$f(z) = Re\{p + \int_0^z \frac{1+a}{1-a}\phi' dz\} + iIm\{\phi\}.$$

□

**Theorem 3.** *If $f = h + \overline{g}$ is of the form* (1.1), *then we have*

$$\phi_1 = h' + g', \ \phi_2 = -i(h' - g'), \ \text{and} \ \phi_3 = 2ibh' \tag{2.2}$$

*where $a = b^2$.*

**Proof.** $Re\{w\} = \frac{w+\overline{w}}{2}$ implies that

$$\begin{aligned} f =& Re\int_0^\zeta \phi_1(z)dz + iRe\int_0^\zeta \phi_2(z)dz + c \\ =& \frac{\int_0^\zeta \phi_1(z)dz + \overline{\int_0^\zeta \phi_1(z)dz}}{2} + i\frac{\int_0^\zeta \phi_2(z)dz + \overline{\int_0^\zeta \phi_2(z)dz}}{2} + c \\ =& \frac{1}{2}\int_0^\zeta (\phi_1 + i\phi_2)dz + \overline{\frac{1}{2}\int_0^\zeta (\phi_1 - i\phi_2)dz} + c. \end{aligned}$$

From $f = h + \overline{g}$, we have

$$h = \frac{1}{2}\int_0^\zeta (\phi_1 + i\phi_2)dz + c, \ g = \frac{1}{2}\int_0^\zeta (\phi_1 - i\phi_2)dz.$$

Hence

$$h' + g' = \frac{1}{2}(\phi_1 + i\phi_2) + \frac{1}{2}(\phi_1 - i\phi_2) = \phi_1.$$

Similarly, we get

$$h' - g' = \frac{1}{2}(\phi_1 + i\phi_2) - \frac{1}{2}(\phi_1 - i\phi_2) = i\phi_2.$$

Therefore $\phi_2 = -i(h' - g')$. Since $\phi_1^2 + \phi_2^2 + \phi_3^2 = 0$, we obtain

$$\phi_3^2 = -(h' + g')^2 - [-i(h' - g')]^2 = -4h'g' = -4h'(ah') = -4ah'^2 \tag{2.3}$$

where $a = g'/h'$. The equation (2.3) tells us that not all function $a$ correspond to nonparametric minimal surfaces. That is, $a$ must be a perfect square. Let $a = b^2$. Then $b$ is analytic in $D$ and $|b| < 1$. Thus $\phi_3 = 2i\sqrt{a}h' = 2ibh'$. □

**Theorem 4.** *The analytic function $\phi = h - g$ is of the form*

$$\phi(z) = \frac{p - \overline{p}z}{1 - z}. \tag{2.4}$$

**Proof.** By Theorem 1, $\phi$ is a conformal univalent mapping from $D$ onto $\Omega$ with $\phi(0) = p$. We begin by finding all Möbius transformations $\phi$ of $D$ onto the upper halfplane $\Omega$ with $\phi(0) = p \in \Omega$, $\phi(e^{i\alpha}) = 0$, $\phi(e^{i\beta}) = \infty$ ; by the simple calculation, we obtain

$$\phi = \frac{p(1 - e^{-i\alpha}z)}{1 - e^{-i\beta}z}$$

where $\alpha \neq \beta$. Since $\phi(e^{i\beta}z) = \frac{p(1-e^{-i\alpha+i\beta}z)}{1-z}$ and a rotation of the disk $D$ is simply reparametrizations of the same surface, it is no loss of generality to assume that $e^{-i\beta} = 1$. So we get $\phi(z) = \frac{p(1-e^{-i\alpha}z)}{1-z}$ with $\phi(0) = p$. Since the boundry values of $\phi$ are on $\partial\Omega$, $Im\{\phi(e^{i(\pi+\frac{\alpha}{2})})\}$ must be 0. From this, we get $e^{i\frac{\alpha}{2}} = p$. Therefore $\phi(z) = \frac{p-\bar{p}z}{1-z}$ with $\phi(0) = p$. □

Now we are ready to find out some conformal univalent harmonic mappings from $D$ onto the upper halfplane $\Omega = \{w : Im\{w\} > 0\}$ that arise in connection with the nonparametric minimal surface $S$ that lie over $\Omega$. Let $b(z) = \pm z$. From (1.1), (2.1), (2.2), and (2.4), we obtain the followings;

$$\begin{aligned}
u &= Re\{p + \int_0^z \frac{1+b^2}{1-b^2}\phi' dz\} \\
&= p_1 - 2p_2 Im \int_0^z \frac{1+z^2}{(1-z^2)(1-z)^2} dz \\
&= p_1 - p_2 Im\{\frac{1}{2} log\frac{1+z}{1-z} + \frac{z}{(1-z)^2}\} \\
&= p_1 + \frac{ip_2}{2}\left[\left(\frac{1}{2}log\frac{1+z}{1-z} + \frac{z}{(1-z)^2}\right) - \overline{\left(\frac{1}{2}log\frac{1+z}{1-z} + \frac{z}{(1-z)^2}\right)}\right], \\
v &= Im\{\phi\} = p_2 Re\{\frac{1+z}{1-z}\} = \frac{p_2}{2}\left[\frac{1+z}{1-z} + \overline{\left(\frac{1+z}{1-z}\right)}\right], \\
F &= Re\int_0^\zeta \phi_3 dz + c_3 = Re\int_0^z 2ibh' dz + c_3 \\
&= Im\int_0^z -2bh' dz + c_3 = Im\int_0^z \frac{-2b(h'-g')}{1-\frac{g'}{h'}} dz + c_3 \\
&= Im\int_0^z \frac{-2b\phi'}{1-b^2} dz + c_3 = \pm 4p_2 Re\int_0^z \frac{z}{(1-z^2)(1-z)^2} dz + c_3 \\
&= \pm p_2 Re\{\frac{z}{(1-z)^2} - \frac{1}{2}log\frac{1+z}{1-z}\} + c_3.
\end{aligned}$$

Let $\frac{1+z}{1-z} = Re^{it}$. Then $R > 0$ and $-\frac{\pi}{2} < t < \frac{\pi}{2}$ because $\frac{1+z}{1-z}$ is a Möbius transformation from $D$ to the right half plane. From these, we get $z = \frac{Re^{it}-1}{Re^{it}+1}$. Substitue this into the above $u$, $v$, and $F$. Then we get the followings;

$$u = p_1 - \frac{p_2}{4}(2t + R^2 sin2t), \quad v = p_2 R cost, \quad F = \pm \frac{p_2}{4}(R^2 cos2t - logR^2) + c_4. \tag{2.5}$$

In this case $u$ varies from $-\infty$ to $\infty$ on each horizontal line. That is, we obtain minimal surfaces over all of $\Omega$.

Finally, we obtain our last result.

**Theorem 5.**

$$f(z) = p_1 + \frac{ip_2}{2}\left[\left(\frac{1}{2}log\frac{1+z}{1-z} + \frac{z}{(1-z)^2}\right) - \overline{\left(\frac{1}{2}log\frac{1+z}{1-z} + \frac{z}{(1-z)^2}\right)} + \frac{1+z}{1-z} + \overline{\left(\frac{1+z}{1-z}\right)}\right]$$

*are harmonic univalent mappings that arise in connection with the minimal surfaces $S$ over $\Omega$ defined by equations* (2.5).

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