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ON UNIFORM CONVERGENCE THEOREMS FOR THE INTERIOR INTEGRAL

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Abstract. In this paper, we introduce interior integral and prove uniform convergence theorems for the interior integral.

1. Introduction

Nowadays our standard integral is the Lebesgue integral that is more powerful than the Riemann integral. But to use the Lebesgue integral we have to study first measure theory that is not easy to understand.

The Riemann integral is easier to learn than the Lebesgue integral. Most of mathematicians ignored the further progress of the Riemann integral.

But two mathematicians, J. Kurzweil and R. Henstock offered an important progress in this direction. They gave an integral that is slight modification of the Riemann integral, i.e., a Riemann type integral([2], [3]).

But the progress is very remarkable. Their integral(called kurzweil–Henstock integral) contains the Lebesgue integral and has good convergence theorems. In recent days. Their integral is comparatively well-known.

In this paper we introduce interior integral and prove uniform convergence theorems for the interior integral.

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2. Interior integral

Assume that $[a,b],[c,d] \subset \mathbb{R}$ are bounded intervals. A set $D = \{[t_{i-1},t_i] : i = 1,2,...,m\}$, where $t_0 = a < t_1 < t_2 < \cdots < t_m = b$, is a *division* of $[a,b]$ if $\bigcup_{i=1}^{m}[t_{i-1},t_i] = [a,b]$.

An *interior tagged* interval $(\tau,[c,d])$ consists of an interval $[c,d] \subset [a,b]$ and a point $\tau \in (c,d)$. Let $P = \{(\tau_j,[c_j,d_j]) : 1 \le j \le m\}$ be a finite collection of non-overlapping interior tagged intervals in $[a,b]$, i.e.,$(c_i,d_i) \cap (c_j,d_j) = \oslash$ if $i \neq j$.

If $\cup_{j=1}^{m}[c_j,d_j] = [a,b]$, then P is called an *interior partition* of $[a,b]$, and if $\bigcup_{j=1}^{m}[c_j,d_j] \subset [a,b]$ then P is called an *interior partial partition* of $[a,b]$.

Let

$$P = \{(\tau_j,J_j) : j = 1,...,k\}, D = \{I_i : i = 1,...,m\}$$

be an interior partition(or, an interior partial partition) and a division of $[a,b]$, respectively. Then we say that P is a *refinement*(or, a *partial refinement*) of D if for every $j = 1,...,k$ there exists an $i = 1,...,m$ such that $J_j \subset I_i$ and we denote $P \ge D$.

In this paper we consider the usual n-dimensional inner product space $\mathbb{R}^n$.

A function $f : [a,b] \longrightarrow \mathbb{R}^n$ is called *regulated* if $f(s+)$ and $f(s-)$ exist for every point s in $[a,b]$.

A function $f : [a,b] \longrightarrow \mathbb{R}^n$ is of *bounded variation* on $[a,b]$ if

$$var_a^b(f) = \sup\left\{\sum_{j=1}^{m} \|f(t_j) - f(t_{j-1})\|\right\} < \infty,$$

where the supremum is taken over all divisions $D = \{[t_{j-1},t_j] : j = 1,2,...,m\}$ of $[a,b]$.

It is well-known fact that if $var_a^b(f) < \infty$ then f is regulated on $[a,b]$.

Let $BV[a,b]$ be the collection of all $\mathbb{R}^n$-valued functions defined on $[a,b]$ of bounded variation. If we let, for $f \in BV[a,b]$,

$$\|f\|_{BV[a,b]} = \|f(a)\| + var_a^b(f)$$

then $(BV[a,b], \|\cdot\|_{BV[a,b]})$ becomes a Banach space.

Let $f : [a,b] \longrightarrow \mathbb{R}^n$ and $\varphi : [a,b] \times [a,b] \longrightarrow \mathbb{R}$. We define $f([c,d]) \equiv f(d) - f(c)$ and $\varphi(\tau,[c,d]) \equiv \varphi(\tau,d) - \varphi(\tau,c)$. For $P = \{(\tau_j,J_j) : 1 \le j \le m\}$ we define $S(d\varphi,P) \equiv \sum_{j=1}^{m} \varphi(\tau_j,J_j)$. Using these notations we introduce an integral.

Definition 2.1. Assume that a functions $\varphi : [a, b] \times [a, b] \longrightarrow \mathbb{R}$ is given. We say that the *interior*(or Dushnik) integral $\int_a^b d\varphi(\tau, t)$ (or shortly, $\int_a^b d\varphi$) exists if there exists a real number L which satisfies that for every $\varepsilon > 0$ there is a division D on $[a, b]$ such that

$$|S(d\varphi, P) - L| < \varepsilon$$

for every interior partition $P = \{(\tau_j, [t_{j-1}, t_j]) : j = 1, 2, ..., m\} \geq D$ of $[a, b]$. We denote $L = \int_a^b d\varphi$.

Let $f, g : [a, b] \longrightarrow \mathbb{R}^n$. In case that $\varphi(\tau, t) = f(\tau)g(t)$(this implies the inner product of $f(\tau)$ and $g(t)$.), we denote

$$\int_a^b d\varphi(\tau, t) = \int_a^b f(s)dg(s)$$

(or shortly, $\int_a^b fdg$). If $\int_a^b f(s)dg(s)$ exists, then we say that fdg is integrable on $[a, b]$.

Theorem 2.1. ([cf.4]) *Assume that $\varphi_1, \varphi_2 : [a, b] \times [a, b] \longrightarrow \mathbb{R}$ are integrable on $[a, b]$. Let $k_1, k_2 \in \mathbb{R}$. Then we have*

$$\int_a^b d(k_1\varphi_1(\tau, t) + k_2\varphi_2(\tau, t)) = k_1 \int_a^b d\varphi_1(\tau, t) + k_2 \int_a^b d\varphi_2(\tau, t).$$

If for $c \in [a, b]$, integrals $\int_a^c d\varphi(\tau, t)$ and $\int_c^b d\varphi(\tau, t)$ exist, then $\int_a^b d\varphi(\tau, t)$ exists also and

$$\int_a^b d\varphi(\tau, t) = \int_a^c d\varphi(\tau, t) + \int_c^b d\varphi(\tau, t).$$

The following statement provides an operative tool in the theory of the interior integral.

Theorem 2.2. *Assume that $\varphi : [a, b] \times [a, b] \longrightarrow \mathbb{R}$. If given $\varepsilon > 0$, there is a division D of $[a, b]$ such that*

$$\left| \sum_{j=1}^{m} \varphi(\tau_j, J_j) - \int_a^b d\varphi(\tau, t) \right| < \varepsilon$$

whenever $P = \{(\tau_j, J_j) : 1 \leq j \leq m\} \geq D$ is an interior partition of $[a, b]$, then for every interior partial partition $P_0 = \{(\tau_i, J_i) : 1 \leq i \leq M\} \geq D$ of $[a, b]$ we have

$$\left| \sum_{i=1}^{M} \left[\varphi(\tau_i, J_i) - \int_{J_i} d\varphi(\tau, t) \right] \right| \leq \varepsilon.$$

and

$$\sum_{i=1}^{M}\left|\varphi(\tau_i, J_i) - \int_{J_i} d\varphi(\tau, t)\right| \leq 2\varepsilon.$$

Proof. Let $\{K_j : 1 \leq j \leq N\}$ be the collection of closed intervals in $[a, b]$ that are contiguous to the intervals of P_0. Let $\eta > 0$. Then there are divisions D_j of K_j for $j = 1, 2, ..., N$ such that

$$\left|S(d\varphi, P_j) - \int_{K_j} d\varphi(\tau, t)\right| < \eta/N$$

whenever $P_j \geq D_j$ is an interior partition of $[a, b]$. We can take $P_j \geq D$, an interior partial partition of $[a, b]$. Fix $\{P_1, P_2, ..., P_N\}$. Let $P = \bigcup_{j=0}^{N} P_j$. Then it is obvious that $P \geq D$ is an interior partition of $[a, b]$.

Then we have

$$\begin{aligned}
&\left|\sum_{i=1}^{M}\left[\varphi(\tau_i, J_i) - \int_{J_i} d\varphi(\tau, t)\right]\right| \\
&= \left|S(d\varphi, P_0) + \sum_{j=1}^{N} S(d\varphi, P_j) - \sum_{i=1}^{M}\int_{J_i} d\varphi(\tau, t) - \sum_{j=1}^{N}\int_{K_j} d\varphi(\tau, t)\right| \\
&+ \left|\sum_{j=1}^{N}(\int_{K_j} d\varphi(\tau, t) - S(d\varphi, P_j))\right| \\
&\leq \left|S(d\varphi, P) - \int_a^b d\varphi(\tau, t)\right| + \sum_{j=1}^{N}\left|S(d\varphi, P_j) - \int_{K_j} d\varphi(\tau, t)\right| \\
&\leq \varepsilon + \eta.
\end{aligned}$$

Since $\eta > 0$ was arbitrary, the first inequality is verified.

It remains to prove the second inequality. Let P_0^1 be the subset of P_0 for which

$$\varphi(\tau_i, J_i) - \int_{J_i} d\varphi(\tau, t) \geq 0$$

and let $P_0^2 = P_0 - P_0^1$. By the first inequality of this lemma, we have

$$\begin{aligned}
&\sum_{i=1}^{M}\left|\varphi(\tau_i, J_j) - \int_{J_i} d\varphi(\tau, t)\right| \\
&= \left|S(d\varphi, P_0^1) - \sum_{(\tau_i,J_i)\in P_0^1} \int_{J_i} d\varphi(\tau, t)\right| \\
&\qquad + \left|S(d\varphi, P_0^2) - \sum_{(\tau_i,J_i)\in P_0^2} \int_{J_i} d\varphi(\tau, t)\right| \\
&\le \varepsilon + \varepsilon = 2\varepsilon.
\end{aligned}$$

This proves the second inequality. □

THEOREM 2.3. *Assume that $f, g : [a, b] \longrightarrow \mathbb{R}^n$ are regulated and that $\int_a^b f dg$ exists. Then we have*

$$\int_a^s f dg = \lim_{\eta\to 0+} \int_a^{s-\eta} f dg + f(s-)\Delta^- g(s) \tag{2.1}$$

and

$$\int_s^b f dg = \lim_{\eta\to 0+} \int_{s+\eta}^{b} f dg + f(s+)\Delta^+ g(s), \tag{2.2}$$

where $\Delta^- g(s) = g(s) - g(s-)$ and $\Delta^+ g(s) = g(s+) - g(s)$.

Proof. Let us first verify (2.1). Similarly we can obtain (2.2). Since $\int_a^s f dg$ exists, for every $\varepsilon > 0$ there exists a division D of $[a, s]$ such that

$$\left|S(f dg, P) - \int_a^s f dg\right| < \varepsilon,$$

whenever $P \ge D$ is an interior partition of $[a, s]$. There exists a positive number η_0 such that $(\tau, [s - \eta, s])$ is an interior partial partition$\ge D$ whenever $0 < \eta < \eta_0$. Then by Theorem 2.3 we have

$$\left|f(\tau)(g(s) - g(s - \eta)) - \int_{s-\eta}^{s} f dg\right| < \varepsilon.$$

Consequently, we get

$$\begin{aligned}
&\left|\int_a^s f dg - \int_a^{s-\eta} f dg - f(\tau)(g(s) - g(s - \eta))\right| \\
&= \left|f(\tau)(g(s) - g(s - \eta)) - \int_{s-\eta}^{s} f dg\right| < \varepsilon.
\end{aligned}$$

If we consider the fact that $s-\eta<\tau<s$, we see that the statement is true. $\square$

3. A uniform convergence theorem for the interior integral

It is known that for any $f \in BV[a,b]$ there exist uniquely determined functions $f^C \in BV[a,b]$ and $f^B \in BV[a,b]$ such that f^C is continuous on $[a,b]$ and f^B is a break function on $[a,b]$ and $f = f^C + f^B$ on $[a,b]$(the Jordan decomposition).

For a set $E \subset [a,b]$,let $\chi_E : [a,b] \longrightarrow \{0,1\}$ be the indicator function on $[a,b]$, i.e., $\chi_E(s) = 1$ if $s \in E$, and $\chi_E(s) = 0$ if $s \notin E$.

If $W = \{w_k\}$ is the set of discontinuities of f in $[a,b]$, then

$$f^B(t) = \sum_{j=1}^{\infty}(\Delta^- f(w_j)\chi_{[w_j,b]}(t) + \Delta^+ f(w_j)\chi_{(w_j,b]}(t)) \quad \text{on} \quad [a,b]. \tag{3.1}$$

where $\Delta^+ f(s) = f(s+) - f(s)$ and $\Delta^- f(s) = f(s) - f(s-)$.

Moreover, if we define

$$f_n^B(t) = \sum_{j=1}^{n}(\Delta^- f(w_j)\chi_{[w_j,b]}(t) + \Delta^+ f(w_j)\chi_{(w_j,b]}(t)) \quad \text{on} \quad [a,b]. \tag{3.2}$$

Then

$$\lim_{n\to\infty} \|f_n^B - f^B\|_{BV} = 0.$$

(cf. e.g, the [5, proof of Lemma I.4.23].)

THEOREM 3.1. *Assume that $f, g_n, g : [a,b] \longrightarrow \mathbb{R}^n$ ($n = 1,2...,$). Suppose that $g_n, g \in BV[a,b]$ for every $n = 1,2,...$ and that*

$$\lim_{n\to\infty} var_a^b(g_n - g) = 0.$$

If $\int_a^b f dg_n$ exists for every $n = 1,2,...$ and f is bounded by some positive number M on $[a,b]$, then $\int_a^b f dg$ exists and we have

$$\int_a^b f dg = \lim_{n\to\infty} \int_a^b f dg_n.$$

Proof. By the definition of the interior integral we have

$$\left| \int_a^b f d(g_n - g_m) \right| \leq M var_a^b(g_n - g_m).$$

Thus by hypotheses the sequence $\left\{\int_a^b f dg_n\right\}$ is a Cauchy sequence. This immediately implies that there is an $L \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \int_a^b f dg_n = L.$$

It remains to show that

$$L = \int_a^b f dg.$$

For a given $\varepsilon > 0$, let n_0 be a positive integer such that

$$\left|\int_a^b f dg_{n_0} - L\right| < \varepsilon \quad \text{and} \quad var_a^b(g_{n_0} - g) < \varepsilon,$$

and let D_ε is a division of $[a,b]$ such that

$$\left|S(f dg_{n_0}, P) - \int_a^b f dg_{n_0}\right| < \varepsilon$$

whenever $P \geq D_\varepsilon$ is an interior partition of $[a,b]$. Hence, given an arbitrary interior partition $P \geq D_\varepsilon$ of $[a,b]$, we have

$$\begin{aligned} |S(f dg, P) - L| \;\leq\; & |S(f dg, P) - S(f dg_{n_0}, P)| \\ & + \left|S(f dg_{n_0}, P) - \int_a^b f dg_{n_0}\right| + \left|\int_a^b f dg_{n_0} - L\right| \\ \leq\; & M var_a^b(g - g_{n_0}) + 2\varepsilon \leq (M+2)\varepsilon. \end{aligned}$$

This completes the proof. $\square$

Theorem 3.2. *Assume that $f : [a,b] \longrightarrow \mathbb{R}^n$ is regulated on $[a,b]$ and that $g \in BV[a,b]$. Then $\int_a^b f dg$ exists and*

$$\begin{aligned} \int_a^b f dg = (R) \int_a^b & f dg^C \\ & + \sum_{j=1}^{\infty} (\Delta^- g(w_j) f(w_j+) + \Delta^+ g(w_j) f(w_j-)) \end{aligned} \tag{3.3}$$

where (R) represents the Riemann–Stieltjes integral and $\{w_j\}$ is the set of discontinuities of g in $[a,b]$.

Proof. The existence of the integral $\int_a^b f dg$ and that $\int_a^b f dg^C = (R) \int_a^b f dg^C$ were verified in [4].

Thus we only show that the second statement.

By the Jordan decomposition of a function of bounded variation, we have $g = g^C + g^B$ on $[a,b]$.

By the definition of g_n^B, we have

$$\int_a^b f dg_n^B$$
$$= \sum_{j=1}^{n} \left[\Delta^- g(w_j) \int_a^b f(s) d\chi_{[w_j,b]}(s) + \Delta^+ g(w_j) \int_a^b f(s) d\chi_{(w_j,b]}(s) \right].$$

Using Theorem 2.4 we have

$$\int_a^b f(s) d\chi_{[w_j,b]}(s) = \int_a^{w_j} f(s) d\chi_{[w_j,b]}(s)$$
$$= \lim_{\eta \to 0+} \int_a^{w_j - \eta} f(s) d\chi_{[w_j,b]}(s) + f(w_j-) \Delta^- \chi_{[w_j,b]}(w_j) = f(w_j-).$$

Similarly we can obtain

$$\int_a^b f(s) d\chi_{(w_j,b]}(s) = f(w_j+).$$

Considering the fact that by Theorem 3.1 we have

$$\lim_{n \to \infty} \int_a^b f dg_n^B = \int_a^b f dg^B,$$

we obtain the required result. □

4. Uniform convergence theorems for three functions

We assume that $(X.\| \cdot \|)$ is a Banach algebra with an identity.

Let for a division $D = \{J_j : 1 \leq j \leq n\}$ of $[a, b]$

$$DSV_a^b(\alpha, D) \equiv \sup\{\| \sum_{j=1}^{n} x_j \alpha(J_j) y_j \|\},$$

where the supremum is taken over all $x_j, y_j \in X$ with $\|x_j\|$ and $\|y_j\| \leq 1$.

A function $\alpha : [a, b] \longrightarrow X$ is *doubly of bounded semi-variation* on $[a, b]$ if

$$\sup_D DSV_a^b(\alpha, D) < \infty,$$

where the supremum is taken over all divisions D of $[a, b]$. We denote $DSV([a, b], X)$ the set of all functions defined on $[a, b]$ of doubly bounded semi-variation.

Let

$$LSV_a^b(\alpha, D) \equiv \sup\{\|\sum_{j=1}^{n} x_j\alpha(J_j)\|\},$$

where the supremum is taken over all $x_j \in X$ with $\|x_j\| \leq 1$.

A function $\alpha : [a,b] \longrightarrow X$ is of *left bounded semi-variation* on $[a,b]$ if

$$\sup_D LSV_a^b(\alpha, D) < \infty,$$

where the supremum is taken over all divisions D of $[a,b]$. We denote $LSV([a,b],X)$ the set of all functions defined on $[a,b]$ of left bounded semi-variation.

Let

$$RSV_a^b(\alpha, D) \equiv \sup\{\|\sum_{j=1}^{n} \alpha(J_j)y_j\|\},$$

where the supremum is taken over all $y_j \in X$ with $\|y_j\| \leq 1$.

A function $\alpha : [a,b] \longrightarrow X$ is of *right bounded semi-variation* on $[a,b]$ if

$$\sup_D RSV_a^b(\alpha, D) < \infty,$$

where the supremum is taken over all divisions D of $[a,b]$. We denote $RSV([a,b],X)$ the set of all functions defined on $[a,b]$ of right bounded semi-variation.

We consider integrals for three functions f, α, g (see, e.g.,[1]). In [1], the authors considered the central Young integral that is also a Riemann type integral.

Definition 4.1. Assume that $f, g, \alpha : [a,b] \longrightarrow X$. We say that interior integral $\int_a^b f(s)\mathrm{d}[\alpha(s)]g(s)$ exists if for every $\varepsilon > 0$ there is a division D on $[a,b]$ such that for

$$S(f\mathrm{d}[\alpha]g, P) \equiv \sum_{j=1}^{m} f(\tau_j)\alpha(J_j)g(\tau_j)$$

we have

$$\|S(f\mathrm{d}[\alpha]g, P) - L\| < \varepsilon$$

for every interior partition $P = \{(\tau_j, J_j) : j = 1,,2,\cdots,m\} \geq D$ of $[a,b]$.

We denote $L = \int_a^b f(s)\mathrm{d}[\alpha(s)]g(s)$ (or shortly $L = \int_a^b f\mathrm{d}[\alpha]g$) (see, [4]).

Let $\|f\|_\infty \equiv \sup_{s\in[a,b]}\|f(s)\|$, where $f:[a,b]\longrightarrow X$ is a bounded function on $[a,b]$.

LEMMA 4.1. *Assume that $f,g,\alpha:[a,b]\longrightarrow X$. Suppose that $\int_a^b f\mathrm{d}[\alpha]g$ exists. Then we have*

$$\left\|\int_a^b f\mathrm{d}[\alpha]g\right\| \leq \|f\|_\infty\cdot\|g\|_\infty\cdot DSV_a^b[\alpha]$$

Proof. Since $\int_a^b f\mathrm{d}[\alpha]g$ exists, there is a division D of $[a,b]$ such that

$$\left\|\int_a^b f\mathrm{d}[\alpha]g - \sum_{j=1}^m f(\tau_j)\alpha(J_j)g(\tau_j)\right\| < \varepsilon.$$

whenever $\{(\tau_j,J_j):1\leq j\leq m\}\geq D$ is an interior partition of $[a,b]$.

Thus

$$\begin{aligned}\left\|\int_a^b f\mathrm{d}[\alpha]g\right\| &\leq \left\|\int_a^b f\mathrm{d}[\alpha]g - \sum_{j=1}^m f(\tau_j)\alpha(J_j)g(\tau_j)\right\| \\ &\quad + \left\|\sum_{j=1}^m f(\tau_j)\alpha(J_j)g(\tau_j)\right\| \\ &\leq \varepsilon + \|f\|_\infty\|g\|_\infty DSV_a^b[\alpha].\end{aligned}$$

Since $\varepsilon>0$ was arbitrary, we obtain the required result. □

THEOREM 4.2. *Assume that $f,g_n,g,\alpha:[a,b]\longrightarrow X$. If*

$$\alpha\in DSV([a,b],X)$$

and $\int_a^b f\mathrm{d}[\alpha]g_n$ exists for every $n=1,2,...$, and the sequence $\{g_n\}$ converges uniformly to g on $[a,b]$. Then integral $\int_a^b f\mathrm{d}[\alpha]g$ exists and

$$\int_a^b f\mathrm{d}[\alpha]g = \lim_{n\to\infty}\int_a^b f\mathrm{d}[\alpha]g_n.$$

Proof. Let $\varepsilon>0$ be given. Since the sequence g_n converges g uniformly on $[a,b]$ there is a positive integer N_1 such that for any $n>N_1$ and $s\in[a,b]$ we have

$$\|g_n(s)-g(s)\|_X < \varepsilon/(\|f\|_\infty+1).$$

By Lemma 4.2, we have

$$\left\|\int_a^b f\mathrm{d}[\alpha]g_n - \int_a^b f\mathrm{d}[\alpha]g_m\right\| = \left\|\int_a^b f\mathrm{d}[\alpha](g_n-g_m)\right\|$$

$$\leq \|f\|_\infty \|g_n - g_m\|_\infty DSV_a^b[\alpha] \leq \varepsilon DSV_a^b[\alpha]$$

for $m, n > N_1$. Since X is a Banach space this inequality implies that the limit

$$\lim_{n\to\infty} \int_a^b f \mathrm{d}[\alpha] g_n = I$$

exists. Let N_2 be a positive number such that

$$\left\| \int_a^b f \mathrm{d}[\alpha] g_m - I \right\| < \varepsilon$$

for $m > N_2$. Let now $m > N = \max(N_1, N_2)$ be fixed. Since the integral $\int_a^b f \mathrm{d}[\alpha] g_m$ exists, there is a division D of $[a, b]$ such that

$$\left\| \sum_{j=1}^k f(\tau_j)\alpha(J_j) g_m(\tau_j) - \int_a^b f \mathrm{d}[\alpha] g_m \right\| < \varepsilon$$

provided $P = \{(\tau_j, J_j) : j = 1, ..., k\} \geq D$ is an interior partition of $[a, b]$. For such a partition P, we have

$$\|S(f\mathrm{d}[\alpha]g, P) - I\| = \left\| \sum_{j=1}^k f(\tau_j)\alpha(J_j)g(\tau_j) - I \right\|$$

$$\leq \left\| \sum_{j=1}^k (f(\tau_j)\alpha(J_j)g(\tau_j) - f(\tau_j)\alpha(J_j)g_m(\tau_j)) \right\|$$

$$+ \left\| \sum_{j=1}^k f(\tau_j)\alpha(J_j)g_m(\tau_j) - \int_a^b f(s)\mathrm{d}[\alpha(s)]g_m(s) \right\| + \left\| \int_a^b f \mathrm{d}[\alpha] g_m - I \right\|$$

$$\leq \|f\|_\infty \cdot \|g - g_m\|_\infty DSV_a^b[\alpha] + \varepsilon + \varepsilon \leq \varepsilon(\|f\|_\infty DSV_a^b[\alpha] + 2).$$

This completes the proof. □

THEOREM 4.3. *Assume that $f, g, \alpha : [a, b] \longrightarrow X$. If f, g, α are regulated on $[a, b]$ and α is doubly bounded semi-variation on $[a, b]$, then $\int_a^b f \mathrm{d}[\alpha] g$ exist*

Proof. By [4], $\int_a^b f \mathrm{d}[\alpha]$ exists since a Banach algebra $(X, \|\cdot\|)$ has an identity. It is obvious that by the definition of the interior integral,

$$\int_a^b f(s)\mathrm{d}[\alpha(s)]\chi_{(c,b]}(s)x = \int_a^b f(s)\mathrm{d}[\alpha(s)]\chi_{[c,b]}(s)x = \int_c^b f \mathrm{d}[\alpha] x.$$

where $c \in [a, b], x \in X$. If we consider the fact that the set of all linear combinations of $\chi_{(c,b]} \cdot x, \chi_{[c,b]} \cdot y (x, y \in X, c \in [a, b])$ forms a dense subset of the set of all regulated functions on $[a, b]$ for the norm $\| \cdot \|_\infty$ and if we apply the uniform convergence theorem, we get the desired result. □

THEOREM 4.4. *Assume that $f, g, \alpha, \alpha_n : [a, b] \longrightarrow X$ $(n = 1, 2, ...)$. If*

$$DSV_a^b[\alpha_n] \leq M, \quad n = 1, 2, ... \tag{4.1}$$

and

$$\lim_{n \to \infty} x\ \alpha_n(s) y = x \alpha(s) y \tag{4.2}$$

for all $x, y \in X$ with $\|x\| \leq 1$ and $\|y\| \leq 1$.

Then

$$DSV_a^b[\alpha] \leq M \tag{4.3}$$

and the integral $\int_a^b f \mathrm{d}[\alpha] g$ exists, and

$$\int_a^b f \mathrm{d}[\alpha] g = \lim_{n \to \infty} \int_a^b f \mathrm{d}[\alpha_n] g.$$

Proof. Let $D = \{[t_{j-1}, t_j] : j = 1, 2, ..., k\}$ be a division of $[a, b]$. Then we have

$$\| \sum_{j=1}^{k} x_j [\alpha(t_j) - \alpha(t_{j-1}] y_j \| \leq \| \sum_{j=1}^{k} x_j [\alpha_n(t_j) - \alpha_n(t_{j-1})] y_j \|$$
$$+ \| \sum_{j=1}^{k} (x_j [\alpha_n(t_j) - \alpha(t_j)] y_j - x_j [\alpha_n(t_{j-1}) - \alpha(t_{j-1})] y_j) \|.$$

By (4.1) the first summand is $\leq M$. By (4.2) for every $j = 1, 2, ..., k$ there exists N_j such that for all $n \geq N$ we have

$$\|x_j [\alpha_n(t_i) - \alpha(t_i)] y_j\| \leq \frac{\varepsilon}{2k},$$

where $i = j, j-1$. Hence for $n \geq \max\{N_1, N_2, ..., N_k\}$ we have

$$\| \sum_{j=1}^{k} x_j [\alpha(t_j) - \alpha(t_{j-1})] y_j \| \leq M + \varepsilon.$$

This proves (4.3).

Let $F_n(g) = \int_a^b f \mathrm{d}[\alpha_n] g$ and $F(g) = \int_a^b f \mathrm{d}[\alpha] g$. Then, by Lemma 4.3, we have

$$\|F_n\| \leq \|f\|_\infty DSV_a^b[\alpha_n] \quad \text{and} \quad \|F\| \leq \|f\|_\infty DSV_a^b[\alpha].$$

Hence $\|F_n\| \le \|f\|_\infty M$ and $\|F\| \le \|f\|_\infty M$.

We have, for $x \in X$ and $c \in [a,b]$, by the definition of the interior integral,

$$\lim_{n\to\infty}\int_a^b f(s)\mathrm{d}[\alpha_n(s)]\chi_{(c,b]}(s)x = \lim_{n\to\infty}\int_a^b f(s)\mathrm{d}[\alpha_n(s)]\chi_{[c,b]}(s)x$$
$$= \lim_{n\to\infty}\int_c^b f(s)\mathrm{d}[\alpha_n(s)]x = \int_c^b f(s)\mathrm{d}[\alpha(s)]x$$
$$= \int_a^b f(s)\mathrm{d}[\alpha(s)]\chi_{[c,b]}(s)x = \int_a^b f(s)\mathrm{d}[\alpha(s)]\chi_{(c,b]}(s)x.$$

This immediately implies that

$$\lim_{n\to\infty}\int_a^b f(s)\mathrm{d}[\alpha_n(s)]g(s) = \int_a^b f(s)\mathrm{d}[\alpha(s)]g(s)$$

for every step-function g on $[a,b]$.

Let g is a regulated function on $[a,b]$. Since the set of all step-functions on $[a,b]$ is dense in the set of all regulated functions on $[a,b]$ for the norm $\|\cdot\|_\infty$, there exists g_ε that is a step-function on $[a,b]$ such that $\|g-g_\varepsilon\|_\infty \le \varepsilon$. Hence

$$\|F(g)-F_n(g)\| \le \|F(g-g_\varepsilon)\| + \|F(g_\varepsilon)-F_n(g_\varepsilon)\| + \|F_n(g-g_\varepsilon)\|$$
$$\le \|F\|\cdot\|g-g_\varepsilon\| + \|F(g_\varepsilon)-F_n(g_\varepsilon)\| + \|F_n\|\cdot\|g-g_\varepsilon\|$$
$$\le \|f\|_\infty M\varepsilon + \varepsilon + \|f\|_\infty M\varepsilon \le \varepsilon(2\|f\|_\infty M + 1),$$

since we just proved that for a step-function g, we have

$$\lim_{n\to\infty}\int_a^b f(s)\mathrm{d}[\alpha_n(s)]g(s) = \int_a^b f(s)\mathrm{d}[\alpha(s)]g(s),$$

there exists a sufficiently large positive integer n such that $\|F(g_\varepsilon) - F_n(g_\varepsilon)\| \le \varepsilon$. This completes the proof. □

References

[1] R. M. Dudley, R. Norvaiša, *Differentiability of six operators on nonsmooth functions and p-variation*, Lecture Notes in Mathematics. 1703. Springer, Berlin(1999)

[2] R. A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron, and Henstock,* Amer. Math. Soc,(1994)

[3] R. Henstock, *Lectures on the Theory of Integration*, World scientific, Singapore, 1988.

[4] C. S. Hönig, *Volterra Stieltjes-integral equations*, North Holland and American Elevier, Mathematics Studies 16, Amsterdam and New York, 1975

[5] S. Schwabik, M. Tvrdý and O. Vejvoda, *Differential and Integral Equations: Boundary Value Problems and Adjoints*, Academia and D, Reidel, Praha and Dordrecht, (1979)

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