

## A REDUCED-ORDER MODELLING FOR ROSENAU-RLW EQUATION WITH B-SPLINE GALERKIN FINITE ELEMENT METHOD\*

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ABSTRACT. We apply a reduced-order method based on B-spline Galerkin finite elements formulation to Rosenau-RLW equation for the first time and explain their process in detail. The ensemble of snapshots is very large generally, and it is difficult to apply POD to the ensemble of snapshots directly. Hence, we try to pick up important snapshots among the whole data. In this paper, we represent three different reduced-order schemes. First, the classical POD technique is examined. Second, (equally sampled snapshots) are exploited for POD technique. Finally, afterward sampling snapshots by CVT, for those snapshots, POD technique is implemented again.

### 1. Introduction

In the study of the dynamics of dense discrete systems, the case of wave-wave and wave-wall interactions cannot be described using the well-known Korteweg-de Vries (KdV) equation which describes nonlinear wave propagation phenomena. To overcome this flaw of the KdV equation, Rosenau [24, 25] proposed the so-called Rosenau equation. On the other hand, as an alternative to the KdV equation the regularized long wave (RLW) equation has been studied by Benjamin et al.[2] and others[3, 14, 15, 16, 18]. Considering the nonlinear wave further in the mathematical model, one can include the viscous term  $-u_{xxt}$  in the

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Rosenau equation, i.e.,

$$u_t + u_{xxxxt} - u_{xxt} + u_x + uu_x = 0.$$

This equation is usually called the Rosenau-RLW equation. The existence and uniqueness of the solution and its error analysis of approximate finite element solution are given by Atouani et al.[1]. Zuo et al.[26] have proposed a Crank-Nicolson finite difference method for the RRLW equation. Pan et al.[17] also have presented a three-level difference scheme for the Rosenau-RLW equation.

One of the main contribution of this work is to attempt to determine accurate approximate solutions of a complex system using very few degrees of freedom based on quadratic B-spline finite element Galerkin approximation for the following Rosenau-RLW equation:

$$(1.1) \quad \begin{cases} u_t + u_{xxxxt} - u_{xxt} + u_x + uu_x = 0, & x \in (0, L), \quad t \in (0, T], \\ u(0, t) = u(L, t) = 0, \quad u_x(0, t) = u_x(L, t) = 0, & t \in (0, T], \\ u(x, 0) = u_0(x), & x \in [0, L]. \end{cases}$$

Spline functions yield smooth interpolating curves which are less likely to exhibit the large oscillation characteristics of high-degree polynomials. This method gives an interpolating polynomial that is smoother and has smaller error than some other interpolating polynomials such as Lagrange polynomial and Newton polynomial. An approximation of the Rosenau-Burgers equation using quadratic B-spline finite element method and related error analysis are studied by Piao et al.[22]. Here, we exploit quadratic B-spline finite element method again to approximate the solution of the Rosenau-RLW equation.

Another important contribution of this work is to develop a reduced-order model for the Rosenau-RLW equation with a new technique dealing with a large snapshot set. When studying turbulent and chaotic systems and in the real-time feedback control of complex systems, one recognizes model reduction improves an efficiency in computation of complex system. For the past decades, the proper orthogonal decomposition (POD) has been one of the most popular reduced-order modeling technique in many studies and it is a main tool for model reduction; see [10, 4, 5, 21]. The POD starts with a set of snapshots that are generated by evaluating the computational solution of transient problems at several instants of time or by evaluating the computational solution for several values of the parameters appearing in the problem description. A combination of the two is sometimes used and the snapshots

must contain sufficient information to accurately represent the dynamics of solutions of the PDEs. Thereafter, the well-known singular value decomposition (SVD) is applied to the snapshot matrices and POD basis is then given by the left singular vectors corresponding to the most dominant singular values of the matrix having the snapshot vectors as its columns. However, the size of the snapshot matrices are sometimes very large and it consumes a lot of time or even not to derive the POD bases directly from those matrices. Thus, some people would like to select a part of the snapshots from the original snapshots set, then construct more smaller size matrix. For such a matrix, one can use the SVD method to get the POD bases. So far, one of the common method is the uniform selection method [4, 5, 13], which is to choose snapshots in equally ordered way. That method brings a quite good result, but it is not the method of optimal selection.

Meanwhile, the centroidal Voronoi tessellation (CVT) has the property of the systematically extracting best representatives in some optimal sense. In CVT sampling, we start with a snapshot set just as is done in a POD-based setting. We then construct a special Voronoi clustering of the snapshot set for which the means of the clusters are also the generators of the corresponding Voronoi clusters, then the generators of the Voronoi clustering make up a new snapshots set. Therefore, we use the compressed snapshots set by CVT for POD method to construct reduced bases to determine a very low-dimensional approximation to the solution of the complex dynamical system. To do so, we must develop basis functions that are in some way closely related to the problem being approximated in in such reduced models. Once a very low-dimensional reduced basis has been determined, we can employ it to solve the complex system by applying a Galerkin method. In general, reduced bases are globally supported so that the discrete systems are dense in low-dimension but the lack of sparsity in the discrete system.

The rest of paper goes as follows. In Section 2, we describe the B-spline finite element approximation of a solution of the Rosenau-RLW equation. In Section 3, we show how CVT bases are defined and constructed and how they are used to determine very low-dimensional approximations. For a comparison reason, we also briefly review POD-based reduced-order bases and its reduced-order approximations. In Section 3, we compute an example to show how snapshots sets can be generated and to compare and contrast the performance of CVT and POD based reduced-order modeling. In Section 4, we apply a CVT based reduced-model to a distributed feedback control problem for the

Rosenau-RLW equation, which is our main purpose. Some numerical results for a distributed feedback control problem are also given.

## 2. Finite element approximation of the Rosenau-RLW equation

Standard Lagrangian finite element basis functions offer only simple  $C^0$ -continuity and therefore they cannot be used for the spatial discretization of the higher-order differential equations (e.g., third-order differential equation or fourth-order differential equation), but the B-spline basis function can at least achieve  $C^1$ -continuous globally, and its basis function is often used to solve the higher order differential equations.

Let us consider the Rosenau-RLW equation with boundary conditions and the initial condition. We use a variational formulation to define a finite element method to approximate (1.1). A variational formulation of the problem (1.1) is as the following: find  $u \in L^2(0, T; H_0^2(0, L))$  such that

$$(2.1) \quad \begin{cases} \int_0^L u_t v dx + \int_0^L u_{xxt} v'' dx + \int_0^L u_{xt} v' dx \\ \quad + \alpha \int_0^L u_x v' dx + \beta \int_0^L u_x v dx + \int_0^L u u_x v dx \\ = \int_0^L f v dx \quad \text{for all } v \in H_0^1(0, L), \\ u(0, x) = u_0(x) \quad \text{in } [0, L], \end{cases}$$

where  $H_0^2 = \{v \in H^2(0, L) : v(0) = v(1) = 0, v'(0) = v'(1) = 0\}$  and  $H^2(0, L) = \{v \in L^2(0, L) : v' \in L^2(0, L), v'' \in L^2(0, L)\}$ . We write the first spatial derivative as " $\frac{d}{dx} = r$ " and the second spatial derivative as " $\frac{d^2}{dx^2} = r'$ " here. We assume that  $u$  is sufficiently smooth in time throughout this paper.

A typical finite element approximation of (2.1) is defined as follows: we first choose conforming finite element subspaces  $V^h \subset H^2(0, L)$  and then define  $V_0^h = V^h \cap H_0^2(0, L)$ . One then seeks  $u^h(t, \cdot) \in V_0^h$  such that

$$(2.2) \quad \left\{ \begin{aligned} & \int_0^L u_t^h v^h dx + \int_0^L u_{xxt}^h (v^h)'' dx + \int_0^L u_{xt}^h (v^h)' dx \\ & \quad + \alpha \int_0^L u_x^h (v^h)' dx + \beta \int_0^L u_x^h v^h dx + \int_0^L u^h u_x^h v^h dx \\ & = \int_0^L f v^h dx \quad \text{for all } v^h \in V_0^h(0, L), \\ & u^h(0, x) = u_0^h(x) \quad \text{in } [0, L], \end{aligned} \right.$$

where  $u_0^h(x) \in V_0^h$  is an approximation, e.g., a projection, of  $u_0(x)$ .

The interval  $[0, L]$  is divided into  $n$  finite elements of equal length  $h$  by the knots  $x_i$  such that  $0 = x_0 < x_1 < \dots < x_n = L$ . The set of splines  $\{\eta_{-1}, \eta_0, \dots, \eta_n\}$  form a basis for functions defined on  $[0, L]$ . Quadratic B-splines  $\eta_i(x)$  with the required properties are defined by [23],

$$\eta_i(x) = \frac{1}{h^2} \begin{cases} (x_{i+2} - x)^2 - 3(x_{i+1} - x)^2 + 3(x_i - x)^2, & [x_{i-1}, x_i], \\ (x_{i+2} - x)^2 - 3(x_{i+1} - x)^2, & [x_i, x_{i+1}], \\ (x_{i+2} - x)^2, & [x_{i+1}, x_{i+2}], \\ 0, & \text{otherwise,} \end{cases}$$

where  $h = x_{i+1} - x_i, i = -1, 0, \dots, n$ .

The quadratic spline and its first derivative vanish outside the interval  $[x_{i-1}, x_{i+2}]$ . Then the spline function values and its first derivative at the knots are given by

$$(2.3) \quad \begin{cases} \eta_i(x_{i-1}) = \eta_i(x_{i+2}) = 0, \quad \eta_i(x_i) = \eta_i(x_{i+1}) = 1; \\ \eta_i'(x_{i-1}) = \eta_i'(x_{i+2}) = 0, \quad \eta_i'(x_i) = \eta_i'(x_{i+1}) = 1. \end{cases}$$

Thus an approximate solution can be written in terms of the quadratic spline functions as

$$(2.4) \quad u^h(x, t) = \sum_{i=-1}^n a_i(t) \eta_i(x),$$

where  $a_i(t)$  are yet undetermined coefficients.

Each spline covers three intervals so that three splines  $\eta_{i-1}(x), \eta_i(x), \eta_{i+1}(x)$  cover each finite element  $[x_i, x_{i+1}]$ . All other splines are zero in this region. Using Eq.(2.4) and spline function properties (2.3), the nodal values of function  $u^h(x, t)$  and its derivative at the knot  $x_i$  and fixed

time  $\tilde{t}$  can be expressed in terms of the coefficients  $a_i(\tilde{t})$  as

$$(2.5) \quad u^h(x_i, \tilde{t}) = a_{i-1}(\tilde{t}) + a_i(\tilde{t}), \quad \left. \frac{\partial u^h(x, \tilde{t})}{\partial x} \right|_{x=x_i} = \frac{2}{h}(a_i(\tilde{t}) - a_{i-1}(\tilde{t})).$$

From (2.5) and homogeneous boundary conditions we get  $a_{-1}(t) = -a_0(t)$  and  $a_n(t) = -a_{n-1}(t)$ . Hence we have

$$(2.6) \quad u^h(x, t) = \sum_{i=0}^{n-1} a_i(t) \xi_i(x),$$

where  $\xi_0(x) = (\eta_0(x) - \eta_{-1}(x))$ ,  $\xi_i(x) = \eta_i(x)$  ( $i = 1, 2, \dots, n-2$ ),  $\xi_{n-1}(x) = \eta_{n-1}(x) - \eta_n(x)$ . Hence  $n$  unknowns  $a_i(t)$  ( $i = 0, 1, \dots, n-1$ ) for every moment of  $t$  must be determined.

According to Galerkin method the test function  $v^h(x)$  in (2.2) is chosen to be  $v_i^h(x) = \xi_i(x)$  ( $i = 0, 1, \dots, n-1$ ). Substituting (2.6) into (2.2) we obtain

$$\left\{ \begin{aligned} & \sum_{i=0}^{n-1} \left( \int_0^L \xi_i \xi_j dx \right) \frac{da_i(t)}{dt} + \sum_{i=0}^{n-1} \left( \int_0^L \xi_i'' \xi_j'' dx \right) \frac{da_i(t)}{dt} \\ & + \sum_{i=0}^{n-1} \left( \int_0^L \xi_i' \xi_j' dx \right) \frac{da_i(t)}{dt} + \alpha \sum_{i=0}^{n-1} \left( \int_0^L \xi_i' \xi_j' dx \right) a_i(t) \\ & + \beta \sum_{i=0}^{n-1} \left( \int_0^L \xi_i' \xi_j dx \right) a_i(t) + \sum_{i=0}^{n-1} \sum_{k=0}^{n-1} \left( \int_0^L \xi_i \xi_k' \xi_j dx \right) a_i(t) a_k(t) \\ & = \int_0^L f \xi_j dx, \\ & \sum_{i=0}^{n-1} \left( \int_0^L \xi_i \xi_j dx \right) a_i(0) = \int_0^L u_0(x) \xi_j dx, \quad j = 0, 1, \dots, n-1, \end{aligned} \right.$$

which can be written in matrix form as

$$(2.7) \quad \begin{cases} (M + W + S) \frac{d\mathbf{a}}{dt} + (\alpha S + \beta C) \mathbf{a} + (\mathbf{a})^T N \mathbf{a} = \mathbf{f}, \\ M \mathbf{a}_0 = \mathbf{u}_0, \end{cases}$$

where  $\mathbf{a} = (a_0, a_1, \dots, a_{n-1})^T$  and  $\mathbf{a}^0 = (a_0^0, a_1^0, \dots, a_{n-1}^0)^T$ . Elements of the  $n \times n$  matrices  $M, W, S, C$ , the  $n \times n \times n$  nonlinear tensor  $N$ ,

and vectors  $\mathbf{f}$  and  $\mathbf{u}_0$  are given by

$$\begin{cases} M_{ij} = \int_0^L \xi_i \xi_j dx, & W_{ij} = \int_0^L \xi_i'' \xi_j'' dx, & S_{ij} = \int_0^L \xi_i' \xi_j' dx, \\ C_{ij} = \int_0^L \xi_i' \xi_j dx, & N_{ijk} = \int_0^L \xi_i \xi_k' \xi_j dx, & f_j = \int_0^L f \xi_j dx, \\ u_j^0 = \int_0^L u_0(x) \xi_j dx, & i, j, k = 0, 1, \dots, n-1. \end{cases}$$

The system of nonlinear ordinary differential equations (2.7) consists of  $n$  equations with  $n$  unknowns. Moreover, the system (2.7) can be written as standard first order nonlinear ordinary differential equations with initial condition because  $M$  and  $(M+W+S)$  are invertible matrices,

$$(2.8) \quad \frac{d\mathbf{a}}{dt} = (M + W + S)^{-1}(\mathbf{f} - (\alpha S + \beta C)\mathbf{a} - (\mathbf{a})^T N \mathbf{a}), \quad \mathbf{a}_0 = \mathbf{u}_0,$$

where, for the sake of simplicity, denoting  $\mathbf{u}_0 = M^{-1}\mathbf{u}_0$  again. The terms in the right hand side of the first equation of the system (2.8) are continuously differentiable with respect to time, furthermore the whole system (2.8) has the only one solution and its solution has equilibrium when forcing term  $f(x, t)$  tends to zero with an infinite time horizon. In conclusion, we take advantage of the Newton method starting from a given initial condition to obtain the equilibrium solution of the system numerically and generate the snapshots i.e. the  $m$  snapshot vectors

$$\mathbf{a} = [ a_0(t_j) \ a_1(t_j) \ \dots \ a_{n-1}(t_j) ]^T, \quad j = 1, \dots, m,$$

which are determined by evaluating the approximate solution of the system (2.8) at  $m$  equally spaced time values  $t_j$  from  $t_1 = 0$  to  $t_m = T$ .

One can recall that ‘nodal value’ means the solution of the differential equations at the knot, and ‘coefficient’ refers to any coefficient appearing in equations (2.5) and (2.6). The property (2.5) will be used in an approximated solution  $u^h$ , so the nodal values of the full-order and reduced-order solutions at a knot  $x_i$  equal the sum of the coefficients at the knots  $x_{i-1}$  and  $x_i$  except the boundary points.

### 3. Reduced-order bases from the snapshot set

#### 3.1. POD reduced-order bases

We briefly describe reduced-order bases using POD method here. Given a discrete set of snapshot vectors  $\tilde{A} = \{\mathbf{a}_j\}_{j=1}^m$  belonging to  $\mathbb{R}^n$ ,

where  $m < n$ , we form the  $n \times m$  snapshot matrix  $A$  whose columns are the snapshot vectors  $\mathbf{y}_m$ :

$$A = (\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_m).$$

Let

$$U^T AV = \begin{pmatrix} \sum & 0 \\ 0 & 0 \end{pmatrix},$$

where  $U$  and  $V$  are  $n \times n$  and  $m \times m$  orthogonal matrices, respectively, and  $\sum = \text{diag}(\sigma_1, \dots, \sigma_{\tilde{m}})$  with  $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\tilde{m}}$  and  $\tilde{m} \leq m$  be the singular value decomposition of  $A$ . Here,  $\tilde{m}$  is the rank of  $A$ , i.e., the dimension of the snapshot set  $\tilde{Y}$ , which would be less than  $m$  whenever the snapshot set is linearly dependent. It is well-known [8] that if

$$U = (\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n) \quad \text{and} \quad V = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_m),$$

then

$$A\mathbf{v}_i = \sigma_i \mathbf{u}_i \quad \text{and} \quad A^T \mathbf{u}_i = \sigma_i \mathbf{v}_i \quad \text{for} \quad i = 1, \dots, \tilde{m}$$

so that also

$$Y^T A\mathbf{v}_i = \sigma_i \mathbf{u}_i \quad \text{and} \quad AA^T \mathbf{u}_i = \sigma_i \mathbf{v}_i \quad \text{for} \quad i = 1, \dots, \tilde{m}$$

so that  $\sigma_i^2$ ,  $i = 1, \dots, \tilde{m}$ , are the nonzero eigenvalues of  $A^T A$  (and also of  $AA^T$ ) arranged in nondecreasing order. Note that the matrix  $C = A^T A$  is simply the correlation matrix for the set of snapshot vectors  $\tilde{A} = \{\mathbf{a}_j\}_{j=1}^m$ , i.e., we have that  $C_{ij} = \mathbf{a}_i^T \mathbf{a}_j$ .

In the reduced-order modeling context, given a set of snapshots  $\tilde{A} = \{\mathbf{a}_j\}_{j=1}^m$  belonging to  $\mathbb{R}^n$ , the POD reduced-basis of dimension  $d \leq m < n$  is the set  $\{\mathbf{w}_k\}_{k=1}^d$  of vectors also belonging to  $\mathbb{R}^n$  consisting of the first  $d$  left singular vectors of the snapshot matrix  $A$ . Thus, one can determine the POD basis by computing the (partial) singular value decomposition of the  $n \times m$  matrix  $A$ . Alternately, one can compute the (partial) eigensystem  $\{\sigma_k^2, \mathbf{v}_k\}_{k=1}^d$  of the  $m \times m$  correlation matrix  $\tilde{C} = A^T A$  and then set  $\mathbf{u}_i = A\mathbf{v}_i$ ,  $k = 1, \dots, d$ .

The  $d$ -dimensional POD basis has the obvious property of orthonormality. It also has several other important properties which we now mention. Let  $\{\mathbf{q}_k\}_{k=1}^d$  be an arbitrary set of  $d$  orthonormal vectors in  $\mathbb{R}^n$  and let  $\text{proj}_S(\mathbf{a})$  denote the projection of a vector  $\mathbf{a} \in \mathbb{R}^n$  onto the set  $S = \text{span}(\{\mathbf{q}_k\}_{k=1}^d)$ . Further, let

$$\mathcal{E}(\mathbf{q}_1, \dots, \mathbf{q}_d) = \sum_{j=1}^n |\mathbf{a}_j - \text{proj}_S(\mathbf{a}_j)|^2,$$

i.e.,  $\mathcal{E}$  is the sum of the squares of the error between each snapshot vector  $\mathbf{a}_j$  and its projection  $\text{proj}_S(\mathbf{a}_j)$  onto the  $S$ . Then, it can be shown that the POD basis  $\{\mathbf{w}_k\}_{k=1}^d$  minimize  $\mathcal{E}(\mathbf{q}_1, \dots, \mathbf{q}_d)$  subject to any given  $d$ -dimensional orthonormal basis  $\{\mathbf{q}_k\}_{k=1}^d$ .

In practical computations of real problems, one may obtain the ensemble of snapshots from physical system, which is drawn from experiments, statistics, and etc. Their sizes of data are usually huge. For example, for weather forecast, one can use the previous weather prediction results to construct the ensemble of snapshots, and then try to restructure the optimal reduced-order basis for the ensemble of snapshots by using the POD method, but he may encounters the difficulty for applying POD to enormous numbers of snapshots directly because of the limit of storage and computational speed. One of the ways to overcome this difficulty is simply to select the snapshot from original massive data in an equally ordered manner. It is, however, not an optimal technique.

Centroidal Voronoi tessellation (CVT) has the property of clustering data set and sampling representatives from that set in some sense of an optimal manner. Du et al.[7] have devised the idea to combine POD and CVT. In this work, we also propose to use CVT to reduce the original snapshot sets, and then apply POD to that set and actually implement a computation for the Rosenau-RLW equation. The mixture of CVT and POD-CVOD method is described in detail below.

### 3.2. CVOD reduced-order bases

The definition of centroidal Voronoi tessellation (CVT) for a snapshot set begins with a set  $\tilde{A} = \{\mathbf{a}_j\}_{j=1}^m$  consisting of  $m$  vectors belonging to  $\mathbb{R}^n$ . The definition also requires us to introduce the notion of clustering or tessellation, which in our context is defined to be a subdivision of the set  $\{V_k\}_{k=1}^l$  is called a clustering or a tessellation of  $\tilde{A}$  if  $V_k \cap V_j = \emptyset$  for  $k \neq j$  and  $\bigcup_{k=1}^l V_k = \tilde{A}$ . Given a set of points  $\{\mathbf{z}_k\}_{k=1}^l$  belonging to  $\mathbb{R}^n$ , the Voronoi region  $V_k$  corresponding to the point  $\mathbf{z}_k$  is defined by

$$V_k = \left\{ \mathbf{a} \in \tilde{A} : |\mathbf{a} - \mathbf{z}_k| < |\mathbf{a} - \mathbf{z}_j|, \text{ for } j = 1, \dots, l, k \neq j \right\}.$$

The points are called *generators*. The set  $\{V_k\}_{k=1}^l$  is a *Voronoi tessellation* or *Voronoi diagram* of  $\tilde{A}$ , and each  $V_k$  is referred to as the *cluster* corresponding to  $\mathbf{z}_k$ . If the cardinality of cluster  $V_k$  is denoted by  $m_k$ ;

clearly,  $\sum_{k=1}^l m_k = m$ , the cardinality of the set  $\tilde{A}$ . The vectors

$$\mathbf{z}_k^* = \frac{1}{m_k} \sum_{\mathbf{a} \in V_k} \mathbf{a}, \quad k = 1, \dots, m$$

are called the *cluster means* or *cluster centroids*. In this situation,  $\mathbf{z}_k^* \neq \mathbf{z}_k$  in general. Given a set  $\tilde{A}$  of  $l$  vectors in  $\mathbb{R}^n$  and positive integer  $l \leq m$ , a *centroidal Voronoi tessellation* (CVT) of  $\tilde{A}$  is a special Voronoi tessellation satisfying

$$\mathbf{z}_k = \mathbf{z}_k^* \quad \text{for } k = 1, \dots, l,$$

i.e., the generators of the Voronoi tessellation coincide with the centroids of the corresponding Voronoi clusters.

CVT have an optimization characterization. Let  $\{\mathbf{z}_k\}_{k=1}^l$  denote an arbitrary set of  $l$  vectors in  $\mathbb{R}^n$  and let  $\{V_k\}_{k=1}^l$  denote a tessellation of the snapshot set  $\tilde{A}$  into  $l$  disjoint subsets and define

$$\mathcal{F}(\mathbf{z}_1, \dots, \mathbf{z}_l; V_1, \dots, V_l) = \sum_{k=1}^l \sum_{\mathbf{a}_m \in V_k} |\mathbf{a}_m - \mathbf{z}_k|^2.$$

Then we refer  $\mathcal{F}$  to as the *energy*. It was proved that a necessary condition for  $\mathcal{F}$  to be minimized holds when  $\{V_k\}_{k=1}^l$  is a CVT of  $\tilde{A}$ ; see [6]. Since, in practice, CVTs can only be approximately constructed, the energy is often used to monitor the quality of the results.

There are several algorithms known for constructing centroidal Voronoi tessellations of a given set. *Lloyd's method* is a deterministic algorithm which is the obvious iteration between computing Voronoi diagrams and mass centroids, i.e., a given set of generators is replaced in an iterative process by the mass centroids of the Voronoi regions corresponding to those generators. *MacQueen's method* is a probabilistic algorithm. There are other probabilistic methods that may be viewed as generalization of both the MacQueen and Lloyd methods and that are plausible to efficient parallelization. To see the detailed algorithms of constructing CVTs, refer to [6, 9, 12, 11, 19, 20].

In the reduced-order modeling context, given a set of snapshots  $\tilde{A}$  belonging to  $\mathbb{R}^n$ , the CVT reduced-basis of dimension  $l < m$  is the set of generators  $\mathbf{z}_k$ ,  $k = 1, \dots, l$ , also belonging to  $\mathbb{R}^n$ , of a CVT of the snapshot set.

For diminishing the original snapshot set, we draw a new snapshot sample by CVT initially, and then apply POD to it for a reduced-order basis. We name this algorithm CVOD from now on.

#### 4. Reduced-order modelling for the Rosenau-RLW equation

For reduced-order modeling applications, the snapshot vectors are usually coefficient vectors in the expansion of the finite element approximation of the solution of the partial differential equation at different moments in time. Thus, to each snapshot vector, there corresponds a finite element basis function.

We now show how a reduced-order basis is used to define a reduced-order model for the stochastic Burgers' equation. For the sake of simplicity, we only discuss the case for which the snapshot set is viewed as a set of finite element coefficient vectors; the case for which the snapshot set is a set of finite element functions proceeds in similar manner.

Let  $\{\mathbf{r}_k\}_{k=1}^d$  be a  $n$  dimensional reduced-order basis (POD basis or CVOD basis) corresponding to the snapshot set  $\{\mathbf{a}_j\}_{j=1}^m$ . For each  $\mathbf{r}_k$ ,  $k = 1, \dots, d$ , there is a finite element function.

$$\psi_k(x) = \sum_{i=1}^n r_{i,k} \xi_i(x) \in V^h$$

where  $w_{i,k}$  denotes the  $i$ th component of  $\{\mathbf{r}_k\}$ . Let

$$V^d = \text{span}(\{\mathbf{r}_k\}_{k=1}^d) \subset V^h.$$

We then seek a reduced basis approximation of the form

$$u^{rom}(x, t) = \sum_{k=1}^d c_k(t) \psi_k(x) \in V^d.$$

where  $c_k(t)$  are unknown and the reduced-order basis functions satisfy homogeneous boundary conditions.

We set  $V_0^d = V^d \cap H_0^2(0, L)$ , then  $V_0^d$  is a  $d$  dimensional subspace of  $V_0^h$  spanned by reduced-order basis. We then determine  $u^{rom}(\cdot, t) \in V^d$  from the discrete problem:

$$\left\{ \begin{array}{l} \int_0^L u_t^{rom} v^d dx + \int_0^L u_{xxt}^{rom} (v^d)'' dx + \int_0^L u_{xt}^{rom} (v^d)' dx \\ \quad + \alpha \int_0^L u_x^{rom} (v^d)' dx + \beta \int_0^L u_x^{rom} v^d dx + \int_0^L u^{rom} u_x^{rom} v^d dx \\ = \int_0^L f v^d dx \quad \text{for all } v^d \in V_0^d(0, L), \\ u^{rom}(0, x) = u_0^d(x) \quad \text{in } [0, L], \end{array} \right.$$

where  $u_0^d(x) \in V_0^d$  is a projection of  $u_0(x)$ .

We, now, have the system of nonlinear ordinary differential equations that determine the coefficient functions  $\{c_k(t)\}_{k=1}^d$  which can be written in matrix form as

$$(4.1) \quad \begin{cases} (\mathbb{M} + \mathbb{W} + \mathbb{S}) \frac{d\mathbf{c}}{dt} + (\alpha\mathbb{S} + \beta\mathbb{C})\mathbf{c} + (\mathbf{c})^T \mathbb{N}\mathbf{c} = \hat{\mathbf{f}}, \\ \mathbb{M}\mathbf{c}_0 = \hat{\mathbf{u}}_0, \end{cases}$$

where  $\mathbf{c} = (c_1, c_2, \dots, c_n)^T$  and  $\mathbf{c}^0 = (c_1^0, c_2^0, \dots, c_n^0)^T$ . We observe matrices  $\mathbb{M}$ ,  $\mathbb{W}$ ,  $\mathbb{S}$ , and  $\mathbb{C}$  are  $d \times d$ , nonlinear tensor  $\mathbb{N}$  is  $d \times d \times d$ , and vectors  $\hat{\mathbf{f}}$  and  $\hat{\mathbf{u}}_0$  are given by

$$(4.2) \quad \begin{cases} \mathbb{M}_{ij} = \int_0^L \psi_i \psi_j dx, & \mathbb{W}_{ij} = \int_0^L \psi_i'' \psi_j'' dx, & \mathbb{S}_{ij} = \int_0^L \psi_i' \psi_j' dx, \\ \mathbb{C}_{ij} = \int_0^L \psi_i' \psi_j dx, & \mathbb{N}_{ijk} = \int_0^L \psi_i \psi_k' \psi_j dx, & \hat{f}_j = \int_0^L f \psi_j dx, \\ \hat{u}_j^0 = \int_0^L u_0(x) \psi_j dx, & i, j, k = 1, 2, \dots, n. \end{cases}$$

The system of nonlinear ordinary differential equations (4.1) consists of  $n$  equations with  $n$  unknowns.

Let  $R \in \mathbb{R}^{n \times d}$  be a matrix of left singular vectors obtained from a snapshot matrix using the SVD. Then we say that equations (4.2) are

$$(4.3) \quad \begin{cases} \mathbb{M} = R^T M R, & \mathbb{W} = R^T W R, & \mathbb{S} = R^T S R, \\ \mathbb{C} = R^T X R, & \mathbb{N} = R^T N(R) R, & \hat{\mathbf{f}} = R^T \mathbf{f}, \\ \hat{\mathbf{u}}_0 = R^T \mathbf{u}_0. \end{cases}$$

We exploit the Newton method to solve system (4.1) again, starting from a given initial condition. In case of the Burgers equation, it is proved the fact that the reduce-order solution converges to the original solution; see [13]. We can show that the Rosenau-RLW equation holds same result.

REMARK 4.1. Recall that if we divide an interval into  $n$  finite elements, then we have  $n + 2$  nodes for the B-spline basis functions. In the actual computation of this paper, however, the snapshots are different from the numerical solutions which is typically mentioned about the reduce-order partial differential equation model, i.e., the snapshots are not ‘nodal values’, but a ‘coefficient’ which is mentioned in Subsection 3.2.

REMARK 4.2. If nodal values is used as a snapshots set and a reduced order basis is obtained from them, it is impossible to derive the matrices and vectors like equations (4.3). Recall that a vector of nodal values has  $n + 2$  dimension thus, the dimension of the derived reduced-order basis is also  $n + 2$ , which does not fit to the size of matrices, nonlinear tensor, and vectors in Section 2. This is why we use ‘coefficients’ as snapshots instead of nodal values.

## 5. Computational experiments

At any given time  $t$ , we define the error  $E(t)$  to measure  $L^2(0, L)$ -norm of difference between the full-order solution and the reduced-order solution, i.e.,

$$E(t) = \left\| u^h - u^{rom} \right\|_2 = \left[ h \sum_{j=1}^n \left| u_j^h - u_j^{rom} \right|^2 \right]^{1/2},$$

where  $u^h$  denotes the approximate solution determined by full-order B-spline basis and  $u_j^h$  are nodal values of a finite element solution  $u^h$ . Similarly  $u^{rom}$  denotes the approximate solution determined by a reduced-order basis and  $u_j^{rom}$  are nodal values of a reduced-order solution  $u^{rom}$ . It is good to notice that there are two respects to this error. One is the fact that the reduced-order model does not exactly recover the information contained in the original snapshot. The other is the fact that the snapshot set itself cannot exactly represent the full finite element solution.

To compare the efficiency and accuracy of the proposed reduced-order modellings in Section 3, the next examples are considered.

EXAMPLE 5.1. Consider the case with a  $\alpha = 0.01$ ,  $\beta = 700$ , final time  $T = 12$ , and an initial condition  $u_0(x) = 16x^2(1-x)^2$  on a domain  $[0, 1]$ , i.e.,  $L = 1$ .

In this example, we first implement the computations for the spatial discretization with grid size  $h = 1/64$  for the finite element method using the B-spline functions and temporal discretization with  $\Delta t = 1/80$ . As a result, we get  $T/\Delta t = 960$  time-snapshots.

In the beginning, we select 64 equally ordered snapshots from the ensemble of snapshots, for example, if we pick the first snapshot, then the 16th snapshot of the old snapshots set becomes the second new snapshot. Next, we choose 64 snapshots from the whole snapshots by CVT

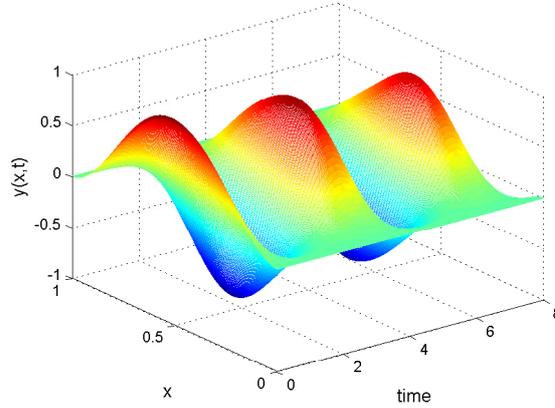


FIGURE 1. Full-order approximate solution of the Rosenau RLW equation for  $\alpha = 0.01$  and  $\beta = 700$  until  $T = 8$

technique. Then the POD method is carried out both new snapshots. We call the former kind of sampling with the POD the uniform POD and one recall that the latter kind of sampling with the POD the CVOD

EXAMPLE 5.2. We variate parameters to construct the reduced-order model. First,  $\alpha = 1$ ,  $\beta = 120$ , second  $\alpha = 0.1$ ,  $\beta = 340$ , and third  $\alpha = 0.01$ ,  $\beta = 700$ . Using the snapshots of three cases, we develop the reduced model to the case for parameters  $\alpha = 0.5$ ,  $\beta = 200$ . Here, a final time is  $T = 3$  and an initial condition is same as Example 5.1.

We compute for the spatial discretization with grid size  $h = 1/64$  for the finite element method using the B-spline functions and temporal discretization with  $\Delta t = 1/80$  again. It is noted that different parameters make an ensemble of snapshots under the same time discretization, i.e., the number of snapshot is  $3 \times (T/\Delta t) = 720$ . The aim of this experiment is to check whether the CVOD scheme works when snapshots obtained from different parameters and we apply to the equation for another parameters.

Once more, we first pick up 64 equally ordered snapshots from the whole snapshots and apply the POD, i.e., the uniform POD. Second we sample 64 snapshots from the old snapshots by CVT technique, i.e., the CVOD. Figure 1 presents the full-order B-spline Galerkin finite element approximate solution of the Rosenau RLW equation of Example 5.1.

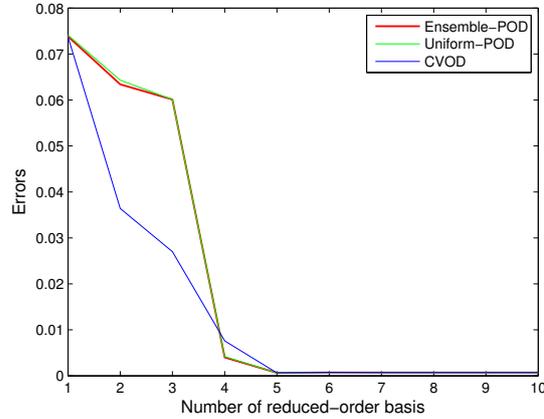


FIGURE 2.  $L^2$ -errors for the solutions of the reduced-order solutions at time  $t = 8$  using time-snapshots over unvaring paramters

#	Ensemble-POD	Uniform-POD	CVOD
1	0.0738	0.0741	0.0737
2	0.0634	0.0643	0.0364
3	0.0601	0.0601	0.0270
4	0.0040	0.0042	0.0076
5	6.7064e-004	6.6844e-004	6.3597e-004
6	7.1450e-004	7.1689e-004	7.7170e-004
7	7.1642e-004	7.1642e-004	7.2081e-004
8	7.1653e-004	7.1653e-004	7.1998e-004
9	7.1653e-004	7.1653e-004	7.1996e-004
10	7.1653e-004	7.1653e-004	7.1996e-004

TABLE 1.  $L^2$ -errors for the solutions of the reduced-order solutions at time  $t = 20$  using time-snapshots over unvaring paramters

We observe that  $L^2$ -error  $E(t)$ s for three different schemes, i.e., the ensemble POD, which treats whole snapshots, the uniform POD, and the CVOD, are decreasing as increasing numbers of the reduces-order bases in Table 1 and Figure 2. If a number of each basis is smaller than five, the CVOD scheme converges more rapidly that other two schemes,

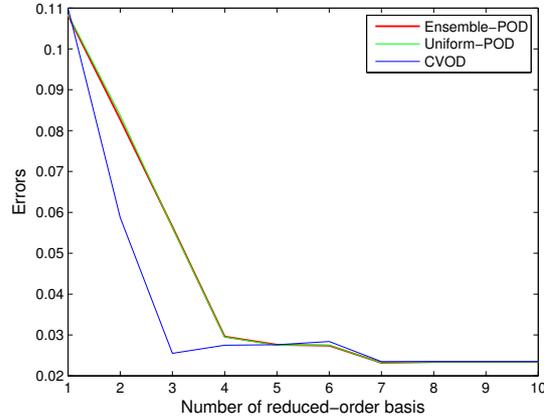


FIGURE 3.  $L^2$ -errors for the solutions of the reduced-order solutions at time  $t = 20$  using time-snapshots over unvaring paramters

#	Ensemble-POD	Uniform-POD	CVOD
1	0.1082	0.1084	0.0737
2	0.0827	0.0837	0.0364
3	0.0565	0.0563	0.0270
4	0.0296	0.0295	0.0076
5	0.0276	0.0276	0.0276
6	0.0274	0.0274	0.0274
7	0.0232	0.0232	0.0232
8	0.0232	0.0232	0.0232
9	0.0232	0.0232	0.0232
10	0.0232	0.0232	0.0232

TABLE 2.  $L^2$ -errors for the solutions of the reduced-order solutions at time  $t = 20$  using time-snapshots over unvaring paramters

but if a number of basis is greater than or equal five, the convergence speed are almost same.

In Table 2 and Figure 3, the graphs show that the error  $E(t)$ s of the predictions in future time  $t > 12$  until  $t = 20$  based on the reduced-order models of the ensemble POD, the uniform POD, and the CVOD

#	Ensemble-POD	Uniform-POD	CVOD
1	0.0619	0.0625	0.0675
2	0.0043	0.0065	0.0018
3	0.0011	8.9819e-004	9.9609e-004
4	6.4738e-004	6.4879e-004	6.4952e-004
5	6.5107e-004	6.6844e-004	6.5108e-004
6	6.5111e-004	6.5112e-004	6.5111 e-004
7	6.5111e-004	6.5111e-004	6.5111e-004
8	6.5111e-004	6.5111e-004	6.5111e-004
9	6.5111e-004	6.5111e-004	6.5111e-004
10	6.5111e-004	6.5111e-004	6.5111e-004

TABLE 3.  $L^2$ -errors for the solutions of the reduced-order solutions at time  $t = 3$  using time-snapshots over varying parameter

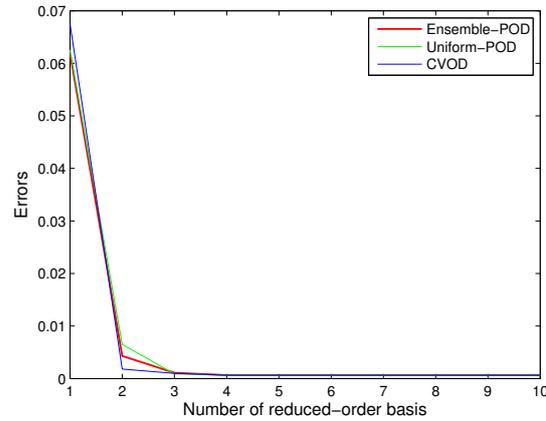


FIGURE 4.  $L^2$ -errors for the solutions of the reduced-order solutions at time  $t = 3$  using time-snapshots over varying parameters

respectly. It exhibits performances of convergence similar to Table 1 and Figure 2. A number of each basis smaller than five, faster convergence of the CVOD scheme appears but a number of basis is greater than or equal seven, the convergence speeds surely look alike.

Even though capturing the snapshots over varying parameters, which cause larger snapshots naturally, in Table 3 and Figure 4, the convergences of the reduced-order solutions to full-order solutions are relatively fast for all techniques and there are not many differences among three schemes.

## 6. Conclusion

An efficient and practical method has been presented for the derivation of low-dimensional approximation of the Rosenau-RLW equation, which is called a reduced-order modelling. One of the most popular method is the POD to apply SVD to an ensemble snapshots directly and we call it ensemble POD here. When one tries model reduction, however, a huge size of snapshot matrix usually troubles him. A simple treatment is to select the snapshots small enough in equally ordered manner, but that scheme involves no optimality. Such a simple technique is called the uniform POD. In this paper, we propose to sample snapshots using the CVT since the generators of the CVT has optimal property in an energy sense. Afterwards choosing snapshots through the CVT, we apply the POD once again. We name such a scheme CVOD.

To obtain snapshots or full-order solutions, we use the quadratic B-spline finite element method to approximate the solution of The Rosenau-RLW equation. Since spline functions gives smoother curves and those are less oscillatory then other higher-degree polynomials, we have smaller error than conventional interpolating polynomials for finite element approximations. In this paper, we give a detailed account of the whole process from how to approximate a full-order the quadratic B-spline finite element solution to how to construct a reduced-order model.

We test the CVOD in two numerical experiments of the Rosenau-RLW equation in comparison with ensemble POD and the uniform POD. The experimental results demonstrate that if a number of reduced-order basis is less than three, then CVOD gives better performance of convergence than other techniques, in contrast, over seven reduced-order basis for each techniques, the three schemes brings almost same results. One of remained question is whether we can analyze a theoretical error of the CVOD and another question is whether the CVOD works well to more complex systems, for example, fluid problems.

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