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WEAK MEASURE EXPANSIVENESS AND CHAOTIC SYSTEMS

Manseob Lee

Abstract. In this article, we consider that if a continuous map $f : X \to X$ of a compact metric space X is Li-York chaotic then it is positively weak measure expansive.

1. Introduction

A chaotic system is an interesting property of topologically dynamical systems. Li-York (1975) defined the mathematical chaos. After, Devaney suggested a chaotic system (see [4]) satisfying (i) a continuous map $f : X \to X$ is transitive, (ii) the periodic point of f is dense in X and (iii) a continuous map $f : X \to X$ is sensitive. Here, a continuous map f is *sensitive* if there is $\epsilon > 0$ such that for any $x \in X$ and $\delta > 0$, there are $y \in X$ with $d(x, y) < \delta$ and $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) > \epsilon$. The notion is related with expansivity. Note that expansivity implies sensitive, but the converse is not true. In fact, the trivial sensitive map is not expansive. It is a motivation of this paper. So, we consider a type of chaos property and a general notion of expansivity.

2. Basic notions

Let X be a compact metric space with metric d and let $f : X \to X$ be a continuous map. For any $x \in X$ and $\delta > 0$, we define the set,

$$\Gamma_\delta(x) = \{y \in X : d(f^i(x), f^i(y)) \le \delta \ \forall i \ge 0\}.$$

It is called the *dynamic δ-ball* of x.

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A continuous map f is said to be *positively expansive* if there is $\delta > 0$ such that $\Gamma_\delta(x) = \{x\}$, for all $x \in X$. For a measure version expansivity, Morales and Sirvent [6] introduced that f is positively measure expansive which is a general concept of expansivity. More detail, let $\mathcal{M}(X)$ be the set of Borel probability measures on X. We say that $\mu \in \mathcal{M}(X)$ is *atomic* if there exists a point $x \in M$ such that $\mu(\{x\}) > 0$. Let $\mathcal{M}^*(X) = \{\mu \in \mathcal{M}(X) : \mu(\{x\}) = 0$ for all $x \in X\}$. A continuous map f is said to be *positively μ expansive* if $\mu(\Gamma_\delta(x)) = 0$ for all $x \in X$ and $\mu \in \mathcal{M}^*(X)$. We say that a continuous map f is *positively measure expansive* if f is positively μ expansive for $\mu \in \mathcal{M}^*(X)$. Recently, Ahn and Lee [2] suggested positively weak measure expansive diffeomorphisms. A continuous map $f : X \to X$ is said to be *positively weak μ expansive* if there is a finite partition $P = \{A_i : i = 1, \dots, n\}$ of X such that $\mu(\Gamma_P(x)) = 0$ for all $x \in X$, where $\Gamma_P(x) = \{y \in X : f^i(y) \in P(f^i(x))$ for all $i \geq 0\}$. Here, $P(x)$ means that there is $A_i \in P$ such that $x \in A_i$. The set $\Gamma_P(x)$ is called the *dynamic P-ball* of x. Note that for a finite partition $P = \{A_i : i = 1, \dots, n\}$, and any $\epsilon > 0$, each A_i is measurable and $\mathrm{diam} A_i \leq \epsilon$. We say that a continuous map f is *positively weak measure expansive* if f is positively weak μ expansive for all $\mu \in \mathcal{M}^*(X)$.

3. Proof of Theorems

For any $\delta > 0$, we say that $S \subset X$ is a *δ-scrambled set* of f if for any different points $x, y \in S$

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 \text{ and } \limsup_{n\to\infty} d(f^n(x), f^n(y)) > \delta. \tag{3.1}$$

We say that f is *Li-Yorke chaotic* ([5]) if f has an uncountable δ-scrambled set for some $\delta > 0$.

A topological space X is said to be *Polish metric* if it is completely separable metric space.

THEOREM 3.1. *Let X be a Polish metric space and $f : X \to X$ be a continuous map. If f has an uncountable δ-scrambled set for some $\delta > 0$ then f is positively weak measure expansive.*

Proof. Since f has an uncountable δ-scrambled set for some $\delta > 0$, by Theorem 16 [3], we can find a closed uncountable δ-scrambled set S. Since $S \subset X$ is closed and X is Polish, we know that S is a Polish metric space with respect to the induced metric. Since S is uncountable, according to [7], there is a non-atomic Borel probability measure ν in S.

For all Borelian $A \subset X$, we define the measure μ such as

$$\mu(A) = \frac{\nu(A \cap S)}{\nu(A)} = \nu(A \cap S).$$

Now, we prove that f is positively weak measure expansive. Let $P = \{A_i : i = 1, \ldots, n\}$ such that $\bigcup_{i=1}^{n} A_i = X$, and diam$A_i \leq \delta/2$. If $x \in S$ and $y \in int(A_i(x)) \cap S$ for some $A_i(x) \in P$, where $A_i(x)$ means that $x \in A_i \in P$ then one can see that $x, y \in S$ and $f^i(y) \in P(f^i(x))$ for all $i \in \mathbb{N} \cup \{0\}$ and so $d(f^i(x), f^i(y)) \leq \delta/2$ for all $i \in \mathbb{N} \cup \{0\}$. Then according to (3.1), we know $x = y$. This implies $A_i \cap S = \{x\}$ for all $x \in S$. Since ν is non-atomic, we have $\mu(\Gamma_P(x)) = \nu(A_i \cap S) = \mu(\{x\}) = 0$, for all $x \in S$. Also, we can easily show that every open set which does not intersect S has μ measure 0, and so, μ is supported in the closure of S. Since $S \subset X$ is closed, one can see that μ is supported on S. According to [1, Theorem 2.1] we have that $\mu(\Gamma_P(x)) = 0$ for μ-a.e $x \in X$. Thus f is positively weak measure expansive. □

Corollary 3.2. *Let X be a compact metric space and let $f : X \to X$ be a continuous map. If f has an uncountable δ-scrambled set for some $\delta > 0$ then f is positively weak measure expansive.*

Proof. Since every compact metric space is Polish metric space, by Theorem 3.1 $f : X \to X$ is positively weak measure expansive. □

Let $I = [0, 1]$ be the interval or C be the unit circle and let $X = I$ or $X = C$.

Theorem 3.3. *Let $f : X \to X$ be a continuous map. If f is Li-York chaotic then f is positively weak measure expansive.*

Proof. Since f is Li-York chaotic map in X, f has an uncountable δ-scrambled set for some $\delta > 0$. According to Theorem 3.1, f is weak measure expansive. □

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Manseb Lee : Department of Mathematics
Mokwon University, Daejeon, 302-729, Korea
E-mail: lmsds@mokwon.ac.kr