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ON SYMMETRIC BI-f-DERIVATIONS OF SUBTRACTION ALGEBRAS

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Abstract. In this paper, we introduce the notion of symmetric bi-f-derivation on subtraction algebra and investigated some related properties. Also, we prove that if $D : X \to X$ is a symmetric bi-f-derivation on X, then D satisfies $D(x - y, z) = D(x, z) - f(y)$ for all $x, y, z \in X$.

1. Introduction

B. M. Schein ([4]) considered systems of the form $(\Phi; \circ, \backslash)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\backslash$" (and hence $(\Phi; \backslash)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka ([6]) discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. In this paper, we introduce the notion of symmetric bi-f-derivation on subtraction algebra and investigated some related properties. Also, we prove that if $D : X \to X$ is a symmetric bi-f-derivation on X, then D satisfies $D(x - y, z) = D(x, z) - f(y)$ for all $x, y, z \in X$.

2. Preliminaries

By a *subtraction algebra* we mean an algebra $(X; -)$ with a single binary operation "$-$" that satisfies the following identities, for any $x, y, z \in X$,

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(S1) $x - (y - x) = x$;
(S2) $x - (x - y) = y - (y - x)$;
(S3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following are true (see [4]).

(p1) $(x - y) - y = x - y$.
(p2) $x - 0 = x$ and $0 - x = 0$.
(p3) $(x - y) - x = 0$.
(p4) $x - (x - y) \leq y$.
(p5) $(x - y) - (y - x) = x - y$.
(p6) $x - (x - (x - y)) = x - y$.
(p7) $(x - y) - (z - y) \leq x - z$.
(p8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.
(p9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.
(p10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$.
(p11) $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$.
(p12) $(x - y) - z = (x - z) - (y - z)$.

for any $x, y, z \in X$,

A mapping d from a subtraction algebra X to a subtraction algebra Y is called a *morphism* if $d(x - y) = d(x) - d(y)$ for all $x, y \in X$. A self map d of a subtraction algebra X which is a morphism is called an *endomorphism*.

LEMMA 2.1. *Let X be a subtraction algebra. Then the following properties hold:*

(1) $x \wedge y = y \wedge x$, *for every* $x, y \in X$.
(2) $x - y \leq x$ *for all* $x, y \in X$.

LEMMA 2.2. *Every subtraction algebra X satisfies the following property*

$$(x - y) - (x - z) \leq z - y$$

for all $x, y, z \in X$.

DEFINITION 2.3. Let X be a subtraction algebra and let Y be a nonempty set of X. Then Y is called a *subalgebra* if $x - y \in Y$ whenever $x, y \in Y$.

DEFINITION 2.4. Let X be a subtraction algebra. A mapping $D(.,.) : X \times X \to X$ is called *symmetric* if $D(x, y) = D(y, x)$ holds for all $x, y \in X$.

DEFINITION 2.5. Let X be a subtraction algebra and $x \in X$. A mapping $d(x) = D(x, x)$ is called *trace* of $D(.,.)$, where $D(.,.) : X \times X \to X$ is a symmetric mapping.

DEFINITION 2.6. Let X be a subtraction algebra and $D : X \times X \to X$ be a symmetric mapping. We call D a symmetric bi-derivation on X if it satisfies the following condition

$$D(x - y, z) = (D(x, z) - y) \wedge (x - D(y, z))$$

for all $x, y, z \in X$.

DEFINITION 2.7. Let X be a subtraction algebra. A function $d : X \to X$ is called an *f-derivation* on X if there exists a function $f : X \to X$ such that

$$d(x - y) = (d(x) - f(y)) \wedge (f(x) - d(y))$$

for all $x, y \in X$.

3. Symmetric bi-f-derivations of subtraction algebras

In what follows, let X denote a subtraction algebra unless otherwise specified.

DEFINITION 3.1. Let X be a subtraction algebra and $D : X \times X \to X$ be a symmetric mapping. We call D a symmetric bi-f-derivation on X if there exists a function $f : X \to X$ such that

$$D(x - y, z) = (D(x, z) - f(y)) \wedge (f(x) - D(y, z))$$

for all $x, y, z \in X$.

Obviously, a symmetric bi-f-derivation D on X satisfies the relation

$$D(x, y - z) = (D(x, y) - f(z)) \wedge (f(y) - D(x, z))$$

for all $x, y, z \in X$.

EXAMPLE 3.2. Let $X = \{0, a, b\}$ be a subtraction algebra with the following Cayley table

$-$	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Define a map $D : X \times X \to X$ by

$$D(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0), (0, a), (a, 0), (0, b), (b, 0), (a, b), (b, a) \\ b & \text{if } (x, y) = (a, a) \\ a & \text{if } (x, y) = (b, b) \end{cases}$$

and $f : X \to X$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ a & \text{if } x = b \\ b & \text{if } x = a \end{cases}$$

Then it is easily checked that D is a symmetric bi-f-derivation of subtraction algebra X. But D is not a symmetric bi-derivation since

$$a = D(b-a, b) \neq (D(b, b)-a) \wedge (b-D(a, b)) = (a-a) \wedge (b-0) = 0 \wedge b = 0.$$

EXAMPLE 3.3. Let $X = \{0, 1, 2, 3\}$ be a set in which "$-$" is defined by

$-$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0

It is easy to check that $(X; -)$ is a subtraction algebra. Define a map $D : X \times X \to X$ by

$$D(x, y) = \begin{cases} 2 & \text{if } (x, y) = (2, 2), (3, 3), (2, 3), (3, 2) \\ 0 & \text{otherwise} \end{cases}$$

and $f : X \to X$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, 1 \\ 2 & \text{if } x = 2, 3 \end{cases}$$

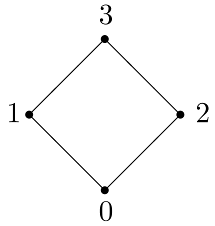

FIGURE 1. Hasse diagram of Example 3.3

Then it is easily checked that D is a symmetric bi-f-derivation of a subtraction algebra X.

PROPOSITION 3.4. *Let X be a subtraction algebra and let D be a symmetric bi-f-derivation on X. Then the following identities hold:*
(1) $D(0, 0) = 0$.
(2) $D(0, x) = 0$ *for all* $x \in X$.
(3) *If* d *is a trace of* D, $d(x) \leq f(x)$ *for all* $x \in X$.

Proof. (1) Since $D(0, 0) = D(0 - 0, 0)$, we have

$$\begin{aligned} D(0,0) = D(0-0,0) &= (D(0,0) - f(0)) \wedge (f(0) - D(0,0)) \\ &= (D(0,0) - f(0)) - ((D(0,0) - f(0)) - (f(0) - D(0,0))) \\ &= (D(0,0) - f(0)) - (D(0,0) - f(0)) = 0. \end{aligned}$$

by (p5).

(2) For all $x \in X$, we get

$$\begin{aligned} D(0,x) =& D(0-0,x) = (D(0,x) - f(0)) \wedge (f(0) - D(0,x)) \\ &= (D(0,x) - f(0)) - ((D(0,x) - f(0)) - (f(0) - D(0,x))) \\ &= (D(0,x) - f(0)) - (D(0,x) - f(0)) = 0. \end{aligned}$$

(3) Since $d(x) = D(x, x)$, we obtain

$$\begin{aligned} d(x) &= D(x,x) = D(x-0,x) = (D(x,x) - f(0)) \wedge (f(x) - D(0,x)) \\ &= (D(x,x) - f(0)) \wedge (f(x) - 0) = (D(x,x) - f(0)) \wedge f(x) \\ &= f(x) \wedge (D(x,x) - f(0)) = f(x) - (f(x) - (D(x,x) - f(0))) \\ &\leq f(x) \quad \text{(by Lemma 2.1 (2))} \end{aligned}$$

□

Proposition 3.5. *Let X be a subtraction algebra and let D be a symmetric bi-f-derivation of X. Then $D(x,y) \leq f(x)$ and $D(x,y) \leq f(y)$ for all $x,y \in X$.*

Proof. For all $x,y \in X$, we have $D(x,y) = D(x-0,y) = (D(x,y) - f(0)) \wedge (f(x) - D(0,y)) = (D(x,y) - f(0)) \wedge f(x) = (D(x,y) - f(0)) - ((D(x,y) - f(0)) - f(x)) = f(x) - (f(x) - (D(x,y) - f(0))) \leq f(x)$. Hence $D(x,y) \leq f(x)$. Similarly, we have $D(x,y) \leq f(y)$.

□

Corollary 3.6. *Let X be a subtraction algebra and let D be a symmetric bi-f-derivation of X. Then $D(x,y) - f(y) \leq f(x) - D(x,y)$ for all $x,y \in X$.*

Proof. For all $x,y \in X$, we have $D(x,y) - f(y) \leq f(x) - f(y)$ and $f(x) - f(y) \leq f(x) - D(x,y)$ from (p6). Hence we obtain $D(x,y) - f(y) \leq f(x) - D(x,y)$. This completes the proof.

□

Theorem 3.7. *Let X be a subtraction algebra and let $D : X \to X$ be a symmetric bi-f-derivation of X. Then D satisfies $D(x-y,z) = D(x,z) - f(y)$ for all $x,y,z \in X$.*

Proof. Let D be a symmetric bi-f-derivation and $x,y,z \in X$. Since $D(x,z) \leq f(x)$ and $D(y,z) \leq f(y)$ by Proposition 3.5, we have

$$D(x,z) - f(y) \leq f(x) - f(y) \leq f(x) - D(y,z)$$

for all $x,y,z \in X$. Hence $D(x-y,z) = (D(x,z) - f(y)) \wedge (f(x) - D(y,z)) = D(x,z) - f(y)$ for all $x,y,z \in X$.

□

Proposition 3.8. *Let D be a symmetric bi-f-derivation of X and let d be a trace of D. Then d is an isotone mapping of X.*

Proof. If $x \leq y$ for all $x,y \in X$, then we have $x = y - w$ for some $w \in X$. Hence

$$\begin{aligned} d(x) = D(x,x) &= D(y-w,y-w) = D(y,y-w) - f(w) \\ &= D(y-w,y) - f(w) = (D(y,y) - f(w)) - f(w) \\ &\leq D(y,y) - f(w) \leq D(y,y) = d(y). \end{aligned}$$

This completes the proof. □

Proposition 3.9. *Let D be a symmetric bi-f-derivation of X and let d be a trace of D. Then $d(x-y) \leq d(x)$ for all $x,y \in X$.*

Proof. Let $x, y \in X$. Then we have

$$\begin{aligned} d(x-y) &= D(x-y, x-y) = D(x, x-y) - f(y) \\ &= D(x-y, x) - f(y) \\ &= (D(x,x) - f(y)) - f(y) \leq D(x,x) = d(x) \end{aligned}$$

This completes the proof. □

PROPOSITION 3.10. *Let D be a symmetric bi-f-derivation of X and let d be a trace of D. If $x \leq y$ for $x, y \in X$, then $d(x \wedge y) = d(x)$.*

Proof. Let $x \leq y$. Then we get $x - y = 0$ and

$$\begin{aligned} d(x \wedge y) &= D(x \wedge y, x \wedge y) \\ &= D(x - (x-y), x - (x-y)) \\ &= D(x - 0, x - 0)) \\ &= D(x, x) = d(x). \end{aligned}$$

This completes the proof.

□

PROPOSITION 3.11. *Let D be a symmetric bi-f-derivation of X and let d be a trace of D. If $f(0) = 0$, we have $d(0) = 0$.*

Proof. Let $x \in X$. Then $d(0) = D(0,0) = D(0-x, 0) = (D(0,0) - f(x)) \wedge (f(0) - D(x,0)) = (d(0) - f(x)) \wedge 0 = 0$.

□

Let X be a subtraction algebra and let D be a symmetric bi-f-derivation of X. For a fixed element $a \in X$, define a mapping $d_a : X \to X$ by $d_a(x) = D(x, a)$ for all $x \in X$.

THEOREM 3.12. *Let X be a subtraction algebra and let D be a symmetric bi-f-derivation of X. Then d_a is a f-derivation of X.*

Proof. Let $x, y \in X$. Then we have

$$\begin{aligned} d_a(x-y) &= D(x-y, a) \\ &= (D(x,a) - f(y)) \wedge (f(x) - D(y,a)) \\ &= (d_a(x) - f(y)) \wedge (f(x) - d_a(y)). \end{aligned}$$

This completes the proof. □

THEOREM 3.13. *Let X be a subtraction algebra and let D be a symmetric bi-f-derivation of X. Then d_a is an isotone mapping of X.*

Proof. Let $x, y \in X$ be such that $x \leq y$. Then we have $x - y = 0$ and so

$$\begin{aligned} d_a(x) &= D(x, a) = D(x - (x - y), a) = D(y - (y - x), a) \\ &= D(y, a) - f(y - x) \leq D(y, a) = d_a(y). \end{aligned}$$

This completes the proof. □

PROPOSITION 3.14. *Let X be a subtraction algebra and let D be a symmetric bi-f-derivation of X. If $x \leq y$ for $x, y \in X$, then $d_a(x \wedge y) = d_a(x)$.*

Proof. Let $x, y \in X$ be such that $x \leq y$. Then we have $x - y = 0$ and so

$$\begin{aligned} d_a(x \wedge y) &= D(x \wedge y, a) \\ &= D(x - (x - y), a) \\ &= D(x, a) = d_a(x). \end{aligned}$$

This completes the proof. □

DEFINITION 3.15. Let X be a subtraction algebra and let D be a symmetric bi-f-derivation of X. If $x \leq w$ implies $D(x, y) \leq D(w, y)$ for $x, y, w \in X$, then D is called an *isotone symmetric bi-f-derivation of X*.

THEOREM 3.16. *Let X be a subtraction algebra and let D be a symmetric bi-f-derivation of X. Then D is an isotone symmetric bi-f-derivation of X.*

Proof. Let $x \leq w$. Then $x = w - v$ from (p8). Hence we have

$$\begin{aligned} D(x, y) &= D(w - v, y) = D(w, y) - f(v) \\ &\leq D(w, y) \end{aligned}$$

This implies that D is an isotone symmetric bi-f-derivation of X.

□

PROPOSITION 3.17. *Let X be a subtraction algebra and let D be an isotone symmetric bi-f-derivation of X. Then the following identities hold.*

(1) $D(x \wedge y, z) \leq D(x, z)$ *for all* $x, y, z \in X$.

(2) $D(x \wedge y, z) \leq D(y, z)$ *for all* $x, y, z \in X$.

Proof. (1) Since $x \wedge y = x - (x - y) \leq x$ from (p4), by Proposition 3.7, we have $D(x \wedge y, z) \leq D(x, z)$ for all $x, y, z \in X$.

(2) Similarly, $x \wedge y = x - (x - y) = y - (y - x) \leq y$ from (p4), we have $D(x \wedge y, z) \leq D(y, z)$ for all $x, y, z \in X$.

□

Let D be a symmetric bi-f-derivation of X. Fix $a \in X$ and define a set $Fix_a(X)$ by

$$Fix_a(X) := \{x \mid D(x, a) = f(x)\}$$

for all $x \in X$.

PROPOSITION 3.18. *Let D be a symmetric bi-f-derivation of X. If f is an endomorphism on X, then $Fix_a(X)$ is a subalgebra of X.*

Proof. Let $x, y \in Fix_a(X)$. Then we have $D(x, a) = f(x)$ and $D(y, a) = f(y)$, and so by Theorem 3.7,

$$\begin{aligned} D(x - y, a) &= D(x, a) - f(y) = f(x) - f(y) \\ &= f(x - y). \end{aligned}$$

Hence we get $x - y \in Fix_a(X)$. This completes the proof.

□

PROPOSITION 3.19. *Let D be a symmetric bi-f-derivation of X and let f be an endomorphism on X. If $x, y \in Fix_a(X)$, we obtain $x \wedge y \in Fix_a(X)$.*

Proof. Let $x, y \in Fix_a(X)$. Then we have $D(x, a) = f(x)$ and $D(y, a) = f(y)$, and so by Theorem 3.7,

$$\begin{aligned} D(x \wedge y, a) &= D(x - (x - y), a) \\ &= D(x, a) - f(x - y) = f(x) - (f(x) - f(y)) \\ &= f(x - (x - y)) = f(x \wedge y). \end{aligned}$$

Hence we get $x \wedge y \in Fix_a(X)$. This completes the proof.

□

DEFINITION 3.20. Let D be a symmetric by-f-derivation of X and let d be a trace of D. Define a set $Kerd$ by

$$Kerd = \{x \in X \mid d(x) = 0\}.$$

PROPOSITION 3.21. *Let D be a symmetric by-f-derivation of X and let d be a trace of D. Then $Kerd$ is a subalgebra of X.*

Proof. Let $x, y \in Kerd$. Then we have $d(x) = D(x, x) = 0$ and $d(y) = D(y, y) = 0$ and so by Theorem 3.7,

$$\begin{aligned} d(x - y) &= D(x - y, x - y) = D(x, x - y) - f(y) \\ &= D(x - y, x) - f(y) = (D(x, x) - f(y)) - f(y) \\ &= 0 - f(y) = 0. \end{aligned}$$

That is, $x - y \in Kerd$. This completes the proof.

□

THEOREM 3.22. *Let D be a symmetric bi-f-derivation of X and let d be a trace of D. If $x \leq y$ and $y \in Kerd$ imply $x \in Kerd$.*

Proof. Let $x \leq y$. Then we have $x - y = 0$ and $d(y) = D(y, y) = 0$. Hence,

$$\begin{aligned} d(x) = D(x,x) &= D(x - (x - y), x - (x - y)) \\ &= D(y - (y - x), y - (y - x)) = D(y, y - (y - x)) - f(y - x) \\ &= D(y - (y - x), y) - f(y - x) = (D(y, y) - f(y - x)) - f(y - x) \\ &= (0 - f(y - x)) - f(y - x) = 0. \end{aligned}$$

That is, $x \in Kerd$. This completes the proof.

□

PROPOSITION 3.23. *Let D be a symmetric bi-f-derivation of X and let d be a trace of D. If $y \in Kerd$ and $x \in X$, then $x \wedge y \in Kerd$.*

Proof. Let $x \in Kerd$. Then we have $d(y) = D(y, y) = 0$. Hence,

$$\begin{aligned} d(x \wedge y) = D(x \wedge y, x \wedge y) &= D(y \wedge x, y \wedge x) \\ &= D(y - (y - x), y - (y - x)) = D(y, y - (y - x)) - f(y - x) \\ &= D(y - (y - x), y) - f(y - x) = (D(y, y) - f(y - x)) - f(y - x) \\ &= (0 - f(y - x)) - f(y - x) = 0. \end{aligned}$$

That is, $x \wedge y \in Kerd$. This completes the proof.

□

DEFINITION 3.24. A nonempty subset I of a subtraction algebra X is called an *ideal* of X if it satisfies

(I1) $0 \in I$,
(I2) for any $x, y \in X$, $y \in I$ and $x - y \in I$ implies $x \in I$.

For an ideal I of a subtraction algebra X, it is clear that $x \leq y$ and $y \in I$ imply $x \in I$ for any $x, y \in X$.

THEOREM 3.25. *Let D be a symmetric by-f-derivation of X and let d be a trace of D. If d is an endomorphism of X, then $Kerd$ is an ideal of X.*

Proof. Let $y \in Kerd$ and $x \in X$ with $x - y \in Kerd$. Then $d(y) = 0$ implies

$$d(x) = d(x) - 0 = d(x) - d(y) = d(x - y) = 0.$$

Hence $x \in Kerd$. This completes the proof.

□

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