

ON AP-HENSTOCK-STIELTJES INTEGRAL FOR FUZZY-NUMBER-VALUED FUNCTIONS ON INFINITE INTERVAL

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ABSTRACT. In this paper we introduce the AP-Henstock-Stieltjes integral for fuzzy-number-valued functions on infinite interval and investigate some properties.

1. Introduction and Preliminaries

It is well-known that the Henstock integral for real valued function was first defined by Henstock [2,3] in 1963. The Henstock integral is more powerful and simpler than the Lebesgue, Feynman integrals.

In 2012, Zengtai Gong and L. Wang introduced the concept of the Henstock-Stieltjes integrals of fuzzy-number-valued functions and obtained some properties([9]).

In this paper we introduce the concept of the AP-Henstock-Stieltjes integral of fuzzy-number-valued functions on infinite interval and investigate some properties.

A Henstock partition of $[a, b]$ is a finite collection $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ such that $\{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ is a non-overlapping family of subintervals of $[a, b]$ covering $[a, b]$ and $\xi_i \in [x_{i-1}, x_i]$ for each $1 \leq i \leq n$. A gauge on $[a, b]$ is a function $\delta : [a, b] \rightarrow (0, \infty)$. A Henstock partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ is said to be δ -fine on $[a, b]$ if $[x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for each $1 \leq i \leq n$.

Let α be an increasing function on $[a, b]$. A function $f : [a, b] \rightarrow R$ is said to be Henstock-Stieltjes integrable to $L \in R$ with respect to α on $[a, b]$ if for every $\epsilon > 0$ there exists a positive function δ on $[a, b]$ such

Received January 14, 2019; Accepted February 01, 2019.

2010 Mathematics Subject Classification: Primary 12A34, 56B34; Secondary 78C34.

Key words and phrases: Fuzzy number, AP-Henstock-Stieltjes Integral .

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that $|\sum_{i=1}^n f(\xi_i)(\alpha(v_i) - \alpha(u_i)) - L| < \epsilon$ whenever $P = \{([u_i, v_i], \xi_i) : 1 \leq i \leq n\}$ is a δ -fine Henstock partition of $[a, b]$. We denote this fact as $(HS) \int_a^b f(x) d\alpha = L$ and $f \in HS[a, b]$. The function f is Henstock-Stieltjes integrable with respect to α on a set $E \subset [a, b]$ if f_{χ_E} is Henstock-Stieltjes integrable with respect to α on $[a, b]$, where χ_E denotes the characteristic function of E .

Fuzzy set $u : R \rightarrow [0, 1]$ is called a fuzzy number if u is a normal, convex fuzzy set, upper semi-continuous and $\text{supp } u = \overline{\{x \in R | u(x) > 0\}}$ is compact. Here $\bar{A}$ denotes the closure of A . We use E^1 to denote the fuzzy number space ([9]).

Let $u, v \in E^1, k \in R$, the addition and scalar multiplication are defined by

$$[u + v]_\lambda = [u]_\lambda + [v]_\lambda, [ku]_\lambda = k[u]_\lambda,$$

where $[u]_\lambda = \{x : u(x) \geq \lambda\} = [u_\lambda^-, u_\lambda^+]$ for any $\lambda \in [0, 1]$.

We use the Hausdorff distance between fuzzy numbers given by $D : E^1 \times E^1 \rightarrow [0, \infty)$ as follows

$$D(u, v) = \sup_{\lambda \in [0, 1]} d([u]_\lambda, [v]_\lambda) = \sup_{\lambda \in [0, 1]} \max\{|u_\lambda^- - v_\lambda^-|, |u_\lambda^+ - v_\lambda^+|\},$$

where d is the Hausdorff metric. $D(u, v)$ is called the distance between u and v .

A function $F : [a, b] \rightarrow E^1$ is said to be bounded if there exists $M \in R$ such that $\|F(x)\| = D(F(x), 0) \leq M$ for any $x \in [a, b]$.

LEMMA 1.1. ([9]) If $u \in E^1$, then

- (1) $[u]_\lambda$ is non-empty bounded closed interval for all $\lambda \in [0, 1]$.
- (2) $[u]_{\lambda_1} \supset [u]_{\lambda_2}$ for any $0 \leq \lambda_1 \leq \lambda_2 \leq 1$.
- (3) for any $\{\lambda_n\}$ converging increasingly to $\lambda \in (0, 1]$,

$$\bigcap_{n=1}^{\infty} [u]_{\lambda_n} = [u]_\lambda.$$

Conversely, if for all $\lambda \in [0, 1]$, there exists $A_\lambda \subset R$ satisfying (1) ~ (3), then there exists a unique $u \in E^1$ such that $[u]_\lambda = A_\lambda, \lambda \in (0, 1]$, and $[u]_0 = \overline{\cup_{\lambda \in (0, 1]} [u]_\lambda} \subset A_0$.

DEFINITION 1.2. ([9]) Let α be an increasing function on $[a, b]$. A fuzzy number-valued function F is Henstock-Stieltjes integrable with respect to α on $[a, b]$ if there exists a fuzzy number $K \in E^1$ such that for every $\epsilon > 0$ there exists a positive function $\delta(x)$ such that

$$D\left(\sum_{i=1}^n F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), K\right) < \epsilon$$

whenever $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ is a δ -fine Henstock partition of $[a, b]$. We denote this fact as $(FHS) \int_a^b F(x) d\alpha = K$ and $(F, \alpha) \in FHS[a, b]$.

In this paper, $\overline{R}$ denoted the extended real number and for the fuzzy number valued function F defined on $[a, \infty]$, we define $F(\infty) = 0$ and $0 \cdot \infty = 0$.

2. The fuzzy number-valued AP-Henstock-Stieltjes Integral

In this section, we will define the fuzzy number-valued AP-Henstock-Stieltjes integral on $[a, \infty]$, which is an extension of the fuzzy number-valued Riemann-Stieltjes integrals on infinite interval and investigate some of their properties.

Let E be a measurable set in $\overline{R}$ and let $x \in \overline{R}$. The density of E at x is defined by

$$d_x E = \lim_{h \rightarrow 0+} \frac{\mu(E \cap (x-h, x+h))}{2h},$$

provided the limit exists. The point x is called a point of density of E if $d_x E = 1$. The E^d represents the set of all $x \in E$ such that x is a point of density of E .

An approximate neighborhood (or ap-nbd) of $x \in [a, b] \subset \overline{R}$ is a measurable set $S_x \subset [a, b]$ containing x as a point of density. For every $x \in E \subset [a, b]$, choose an ap-nbd $S_x \subset [a, b]$ of x and S_x is finite measurable set if $x \in R$. then we say that $S = \{S_x : x \in E\}$ is a choice on E . A tagged interval $([u, v], x)$ is said to be fine to the choice $S = \{S_x\}$ if $u, v \in S_x$. Let $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ be a finite collection of non-overlapping tagged intervals. If all the members of $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ is fine to a choice S , then we say that P is S -fine. Let $E \subset [a, \infty]$. If P is S -fine and each $\xi_i \in E$, then P is called S -fine on E . If $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ is S -fine and $[a, \infty] = \cup_{i=1}^n [x_{i-1}, x_i]$, then we say that P is S -fine partition of $[a, \infty]$. Note that if $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ is S -fine partition of $[a, \infty]$, then $\xi_n = \infty$ and $[x_{n-1}, x_n]$ is infinite interval in $[a, \infty]$.

DEFINITION 2.1. ([8]) Let α be an increasing function on $[a, b]$. A fuzzy number-valued function F is AP-Henstock-Stieltjes integrable with

respect to α on $[a, b]$ if there exists a fuzzy number $K \in E^1$ such that for every $\epsilon > 0$ there exists a choice S on $[a, b]$ such that

$$D\left(\sum_{i=1}^n F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), K\right) < \epsilon$$

whenever $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ is a S -fine Henstock partition of $[a, b]$.

DEFINITION 2.2. Let α be an increasing function on $[a, \infty]$. A fuzzy number-valued function F is AP-Henstock-Stieltjes integrable with respect to α on $[a, \infty]$ if there exists a fuzzy number $K \in E^1$ and $T \in R^+$ such that for every $\epsilon > 0$ there exists a choice S on $[a, \infty]$ such that

$$D\left(\sum_{i=1}^n F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), K\right) < \epsilon$$

whenever $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ is a S -fine Henstock partition on $[a, b]$ of a finite interval $[a, b] \supset [a, \infty] \cap [-T, T]$. We write $(APFHS) \int_a^\infty F(x) d\alpha = K$ and $(F, \alpha) \in APFHS[a, \infty]$.

The definition of $(F, \alpha) \in APFHS[-\infty, a]$ is similar. Naturally $(F, \alpha) \in APFHS[-\infty, \infty]$ if and only if $(F, \alpha) \in APFHS[-\infty, a]$ and $(F, \alpha) \in APFHS[a, \infty]$, and then

$$\begin{aligned} (APFHS) \int_{-\infty}^\infty F(x) d\alpha \\ = (APFHS) \int_{-\infty}^a F(x) d\alpha + (APFHS) \int_a^\infty F(x) d\alpha. \end{aligned}$$

The fuzzy number-valued function F is AP-Henstock-Stieltjes integrable with respect to α on a set $E \subset [a, \infty]$ if F_{χ_E} is AP-Henstock-Stieltjes integrable with respect to α on $[a, \infty]$, where χ_E denotes the characteristic function of E .

REMARK 2.3. If $F \in APFHS[a, \infty]$, then the integral value is unique.

From the definition of the fuzzy number-valued AP-Henstock-Stieltjes integral and the fact that (E^1, D) is a complete metric space, we can easily obtain the following theorem.

THEOREM 2.4. Let α be an increasing function on $[a, \infty]$. A fuzzy number-valued function F is AP-Henstock-Stieltjes integrable with respect to α on $[a, \infty]$ if and only if for every $\epsilon > 0$ there exists a choice

S on $[a, \infty]$ such that for any S -fine partitions $P = \{([u, v], \xi)\}$ and $P' = \{([u', v'], \xi')\}$, we have

$$D\left(\sum_P F(\xi)(\alpha(v) - \alpha(u)), \sum_{P'} F(\xi')(\alpha(v') - \alpha(u'))\right) < \epsilon.$$

THEOREM 2.5. *Let α be an increasing function on $[a, \infty]$ and let F be a fuzzy number-valued function on $[a, \infty]$. Then the following statements are equivalent:*

- (1) $(F, \alpha) \in APFHS[a, \infty]$ and $(APFHS) \int_a^\infty F(x) d\alpha = K$.
- (2) For any $b > a$, $(F, \alpha) \in APFHS[a, b]$ and $\lim_{b \rightarrow \infty} (APFHS) \int_a^b F(x) d\alpha$ exists and then $\lim_{b \rightarrow \infty} (APFHS) \int_a^b F(x) d\alpha = (APFHS) \int_a^\infty F(x) d\alpha$.

Proof. (1) implies (2) : Since $(APFHS) \int_a^\infty F(x) d\alpha = K$, for any $\epsilon > 0$, there exists a choice S_1 on $[a, \infty]$ and $T \in R^+$ such that for any S_1 -fine partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ on $[a, b]$ of a finite interval $[a, b] \supset [a, \infty] \cap [-T, T]$, we have

$$(2.1) \quad D\left(\sum_{i=1}^n F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), K\right) < \epsilon$$

For $b > T$, we find a choice S_2 on $[a, b]$ such that for any S_2 -fine partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ on $[a, b]$

$$(2.2) \quad D\left(\sum_{i=1}^n F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), \int_a^b F(x) d\alpha\right) < \epsilon$$

Define $S = S_1 \cap S_2$ on $[a, b]$ and let P be S -fine partition on $[a, b]$. then by (2.1) and (2.2)

$$D\left((APFHS) \int_a^b F(x) d\alpha, (APFHS) \int_a^\infty F(x) d\alpha\right) < 2\epsilon.$$

Hence, $\lim_{b \rightarrow \infty} (APFHS) \int_a^b F(x) d\alpha = (APFHS) \int_a^\infty F(x) d\alpha$.

(2) implies (1) : Let $\lim_{b \rightarrow \infty} (APFHS) \int_a^b F(x) d\alpha = L$. Then for any $\epsilon > 0$, there exists $T \in R^+$ such that

$$(2.3) \quad D\left(\sum_{i=1}^n F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), L\right) < \frac{\epsilon}{2}$$

for $b > T$. Let $\{b_n\}$ be a strictly increasing sequence with $b_1 = a$ and $\lim_{n \rightarrow \infty} b_n = \infty$.

For $\epsilon > 0$ there exists a choice S^n on $[b_n, b_{n+1}]$ such that for any S^n -fine partition P_n on $[b_n, b_{n+1}]$

$$(2.4) \quad D\left(\sum_{P_n} F(\xi)(\alpha(v) - \alpha(u)), (APFHS) \int_{b_n}^{b_{n+1}} F(x) d\alpha\right) < \frac{\epsilon}{2^{n+2}}.$$

Define $S = \{S_x : x \in [a, \infty]\}$ by $S_\xi \in S^n$ if $\xi \in [b_n, b_{n+1}]$, $S_\xi \in (b_n, b_{n+1})$ if $\xi \in (b_n, b_{n+1})$ and $S_\infty = (T, \infty]$ if $\xi = \infty$.

Let $b > T$. Since $\lim_{n \rightarrow \infty} b_n = \infty$, there exists N such that $b_N < b \leq b_{N+1}$. Let $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ be a S -fine partition on $[a, b]$. Note that the partition P anchors on all b_n with $n \leq N$ and

$$(2.5) \quad \sum_P F\chi_{[a,b]}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) = \sum_1^{N-1} \sum_P F\chi_{[b_n, b_{n+1}]}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) \\ + \sum_P F\chi_{[b_N, b]}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})).$$

By (2.4),

$$(2.6) \quad D\left(\sum_P F\chi_{[b_n, b_{n+1}]}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), (APFHS) \int_{b_n}^{b_{n+1}} F(x) d\alpha\right) < \frac{\epsilon}{2^{n+2}}.$$

By (2.3), (2.5), (2.6) and Saks-Henstock lemma (lemma 2.8.1 in [7])

$$D\left(\sum_{i=1}^n F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), L\right) < \frac{\epsilon}{2} + \frac{\epsilon}{4} \sum_1^{N-1} \frac{1}{2^n} \\ + D\left(\sum_P F\chi_{[b_N, b]}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), (APFHS) \int_{b_N}^b F(x) d\alpha\right) < \epsilon.$$

Hence, $(F, \alpha) \in APFHS[a, \infty]$. $\square$

THEOREM 2.6. *Let α be an increasing function on $[a, \infty]$ and let F be a fuzzy number-valued function on $[a, \infty]$. Then the following statements are equivalent:*

- (1) $(F, \alpha) \in APFHS[a, \infty]$ and $(APFHS) \int_a^\infty F(x) d\alpha = K$.
- (2) For any $\lambda \in [0, 1]$, F_λ^- and F_λ^+ are AP-Henstock-Stieltjes integrable functions with respect to α on $[a, \infty]$ for $\lambda \in [0, 1]$ uniformly (Choice S is independent of $\lambda \in [0, 1]$) and

$$[(APFHS) \int_a^\infty F(x) d\alpha]_\lambda$$

$$= [(APHS) \int_a^\infty F_\lambda^-(x) d\alpha, (APHS) \int_a^\infty F_\lambda^+(x) d\alpha].$$

Proof. (1) implies (2) : Since $(APFHS) \int_a^\infty F(x) d\alpha = K$, for any $\epsilon > 0$, there exists a choice S on $[a, \infty]$ and $T \in R^+$ such that for any S -fine partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ on $[a, b]$ of a finite interval $[a, b] \supset [a, \infty] \cap [-T, T]$, we have

$$D\left(\sum_{i=1}^n F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), K\right) < \epsilon$$

By definition of D , for any S -fine partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ on $[a, b]$

$$\begin{aligned} \sup_{\lambda \in [0,1]} \max \left\{ \left| \left[\sum_{i=1}^n F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1}))) \right]_\lambda^- - K_\lambda^- \right|, \right. \\ \left. \left| \left[\sum_{i=1}^n F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1}))) \right]_\lambda^+ - K_\lambda^+ \right| \right\} < \epsilon. \end{aligned}$$

Hence, we have

$$\begin{aligned} \left| \sum_{i=1}^n F_\lambda^-(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) - K_\lambda^- \right| < \epsilon, \\ \left| \sum_{i=1}^n F_\lambda^+(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) - K_\lambda^+ \right| < \epsilon. \end{aligned}$$

Therefore, $F_\lambda^-(x)$ and $F_\lambda^+(x)$ are AP-Henstock-Stieltjes with respect to α on $[a, \infty]$ for any $\lambda \in [0, 1]$, and

$$(APHS) \int_a^\infty F_\lambda^-(x) d\alpha = K_\lambda^-, \quad (APHS) \int_a^\infty F_\lambda^+(x) d\alpha = K_\lambda^+.$$

(2) implies (1) : Since $F_\lambda^-(x)$ and $F_\lambda^+(x)$ are AP-Henstock-Stieltjes integrable functions with respect to α on $[a, \infty]$ for any $\lambda \in [0, 1]$, for $\epsilon > 0$, there exists a choice S on $[a, \infty]$ and $T \in R^+$ such that for any S -fine partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ on $[a, b]$ of a finite interval $[a, b] \supset [a, \infty] \cap [-T, T]$ and for any $\lambda \in [0, 1]$, we have

$$\begin{aligned} \left| \sum_{i=1}^n F_\lambda^-(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) - K_\lambda^- \right| < \epsilon, \\ \left| \sum_{i=1}^n F_\lambda^+(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) - K_\lambda^+ \right| < \epsilon. \end{aligned}$$

Then we can prove that $\{[K_\lambda^-, K_\lambda^+] : \lambda \in [0, 1]\}$ satisfies the conditions of Lemma 1.1. By the Lemma 1.1, $\{[K_\lambda^-, K_\lambda^+] : \lambda \in [0, 1]\}$ determines a fuzzy number K . By the definition of D , for any S -fine partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ on $[a, b]$, we obtain

$$D\left(\sum_{i=1}^n F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), K\right) < 2\epsilon$$

Thus, $(F, \alpha) \in APFHS[a, \infty]$ and $(APFHS) \int_a^\infty F(x) d\alpha = K$. $\square$

From theorem 2.6, the following theorem is obvious.

THEOREM 2.7. *Let $F, G \in APFHS[a, \infty]$ and $\beta, \gamma \in R$. Then $\beta F + \gamma G \in APFHS[a, \infty]$ and*

$$(APFHS) \int_a^\infty (\beta F + \gamma G) d\alpha = \beta (APFHS) \int_a^\infty F d\alpha + \gamma (APFHS) \int_a^\infty G d\alpha.$$

THEOREM 2.8. *Let $F, G \in APFHS[a, \infty]$ and $D(F, G)$ is Lebesgue-Stieltjes integrable on $[a, \infty]$. Then*

$$D((APFHS) \int_a^\infty F d\alpha, (APFHS) \int_a^\infty G d\alpha) \leq (L) \int_a^\infty D(F, G) d\alpha.$$

Proof. By definition of distance,

$$\begin{aligned} & D((APFHS) \int_a^\infty F d\alpha, (APFHS) \int_a^\infty G d\alpha) \\ &= \sup_{\lambda \in [0, 1]} \max \left(\left| \left[\left((APHS) \int_a^\infty F d\alpha \right)_\lambda^- \right] - \left[\left((APHS) \int_a^\infty G d\alpha \right)_\lambda^- \right] \right|, \right. \\ & \quad \left. \left| \left[\left((APHS) \int_a^\infty F d\alpha \right)_\lambda^+ \right] - \left[\left((APHS) \int_a^\infty G d\alpha \right)_\lambda^+ \right] \right| \right) \\ &= \sup_{\lambda \in [0, 1]} \max \left(\left| (APHS) \int_a^\infty ([F]_\lambda^- - [G]_\lambda^-) d\alpha \right|, \right. \\ & \quad \left. \left| (APHS) \int_a^\infty ([F]_\lambda^+ - [G]_\lambda^+) d\alpha \right| \right) \\ &\leq \sup_{\lambda \in [0, 1]} \max \left((L) \int_a^\infty |[F]_\lambda^- - [G]_\lambda^-| d\alpha, (L) \int_a^\infty |[F]_\lambda^+ - [G]_\lambda^+| d\alpha \right) \\ &\leq (LS) \int_a^\infty D(F, G) d\alpha. \end{aligned}$$

$\square$

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