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GENERALIZED CUBIC FUNCTIONS ON A QUASI-FUZZY NORMED SPACE

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Abstract. We introduce a generalized cubic functional equation and investigate the Hyers-Ulam stability of the cubic functions as solutions to the generalized cubic functional equation on a quasi-fuzzy anti-β-Banach space by both the direct method and the fixed point method.

1. Introduction

In 1965 Zadeh [43] introduced the theory of fuzzy sets. After his pioneering work the fuzzy theory has become an active area of research and a lot of developments have been made in the fuzzy-set theory in a variety of scientific fields such as computer science, physics, engineering, and even economics, not only psychological and social sciences; see [44, 28]. Katsara [20] in 1984 defined a fuzzy norm on a linear space to construct a fuzzy vector topological structure on the space. Wu and and Fang [39] also introduced a notion of fuzzy normed space independently at the same year and gave the generalization of Kolmogorov's normalized theorem for a fuzzy topological linear space. In 1991 Biswas [2] defined fuzzy inner product spaces in a linear space and fuzzy orthogonality of one element to the other showing that fuzzy intersection of two fuzzy inner product spaces is also a fuzzy inner product space. Since then fuzzy norms and metrics on linear spaces have been defined and studied from a variety of perspectives; see [13, 22, 36, 41]. Cheng and Mordeson [8] in 1994 defined a fuzzy norm on a linear space so that the induced metric is the fuzzy metric introduced by Kramosil and Michalek [21]. In 2003 Bag and Samanta [6] modified the definition of fuzzy norm of Cheng

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and Mordeson type (see again [8]) by removing a regular condition. We will revisit the notion of fuzzy normed spaces just after we introduce the stability problems of functional equations.

In a talk before the Mathematics Club of the University of Wisconsin in 1940, a Polish-American mathematician, S. M. Ulam [38] proposed the stability problem of the linear functional equation $f(x+y) = f(x)+f(y)$ where any solution $f(x)$ of this equation is called a linear function.
To make the statement of the problem precise, *let G_1 be a group and G_2 a metric group with the metric $d(\cdot,\cdot)$. Then given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $f : G_1 \to G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $F : G_1 \to G_2$ with $d(f(x), F(x)) < \epsilon$ for all $x \in G_1$?.* In other words, the question would be generalized as "Under what conditions a mathematical object satisfying a certain property approximately must be close to an object satisfying the property exactly?".
In 1941, the first, affirmative, and partial solution to Ulam's question was provided by D. H. Hyers [16]. In his celebrated theorem Hyers explicitly constructed the linear function (or additive function) in Banach spaces directly from a given approximate function satisfying the well-known weak Hyers inequality with a positive constant. The Hyers stability result was first generalized in the stability of additive mappings by Aoki [1] allowing the Cauchy difference to become unbounded. In 1978 Th. M. Rassias [29] also provided a generalization of Hyers' theorem with the possibly unbounded Cauchy difference for linear mappings. For the last decades, stability problems of various functional equations, not only linear case, have been extensively investigated and generalized by many mathematicians (see [5, 11, 14, 17, 30, 33, 34]).

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

is called a quadratic functional equation and every solution of this functional equation is said to be a quadratic function or mapping (e.g. $f(x) = cx^2$). The Hyers-Ulam stability problem for the quadratic functional equation was first studied by Skof [37] in a normed space as the domain of a mapping of the equation. Cholewa [10] noticed that the results of Skof still hold in abelian groups. In [11] Czerwik obtained the Hyers-Ulam-Rassias stability of the quadratic functional equation. See [3, 27, 42] for more results on the equation (1.1).

Jun and Kim [17] considered the following cubic functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \tag{1.2}$$

since it should be easy to see that a function $f(x) = cx^3$ is a solution of the equation (1.2) as the quadratic equation case. In a year they [18] proved the generalized Hyers-Ulam-Rassias stability of a different version of a cubic functional equation

$$f(x+2y) + f(x-2y) + 6f(x) = f(x+y) + 4f(x-y). \tag{1.3}$$

Since then the stability of cubic functional equations has been investigated by a number of authors (see [9, 26] for details). In particular, Najati [26] investigated the following generalized cubic functional equation

$$f(sx+y) + f(sx-y) = sf(x+y) + sf(x-y) + 2(s^3 - s)f(x) \tag{1.4}$$

for a positive integer $s \geq 2$.

As we notice there are various definitions for the stability of the cubic functional equations and we shall use the following definition of a new type of generalized cubic functional equation

$$\begin{aligned} &f(ax+y) - f(x+ay) + a(a-1)f(x-y) \\ &= (a-1)(a+1)^2 f(x) - (a-1)(a+1)^2 f(y) \end{aligned} \tag{1.5}$$

for all $a \in \mathbb{Z}$ ($a \neq 0, \pm 1$) and investigate the generalized Hyers-Ulam stability problem of the equation (1.5).

The study of fuzzy stability of functional equations can be found in the papers by Mirmostafaee, Moslehian, and Mirzavaziri [24, 25]. In 2010 Jebril and Samanta [19] introduced a fuzzy anti-norm linear space which depends on the idea of fuzzy norm of Bag and Samanta [7] and investigated its important properties. Let us give the definition of fuzzy anti-normed spaces here to define a quasi-fuzzy anti-β-norm with which we will establish a fuzzy version of Hyers-Ulam-Rassias stability in a fuzzy linear space.

Definition 1.1. Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a *fuzzy anti-norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

1. $N(x,t) = 1$ for $t \leq 0$
2. $N(x,t) = 0$ if and only if $x = 0$ for all $t > 0$
3. $N(cx,t) = N(x, \frac{t}{|c|})$ for $c \neq 0$
4. $N(x+y, s+t) \leq \max\{N(x,s), N(y,t)\}$
5. $N(x,t)$ is a non-increasing function of $t \in \mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 0$,
6. for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is said to be a *fuzzy anti-normed linear space*. One may regard the norm $N(x, t)$ as the truth value of the statement of the norm of x is greater than or equal to the real number t. The property (3) in the definition implies that $N(-x, t) = N(x, t)$ for all $t > 0$. Also it is not hard to show that the condition (4) is equivalent to $N(x+y, t) \leq \max\{N(x, t), N(y, t)\}$ for all $x, y \in X$ and $t \in \mathbb{R}$ and hence we easily conclude the triangle inequality

$$N(x + y, t) \leq N(x, t) + N(y, t) \tag{1.6}$$

for all $x, y \in X$ and $t \in \mathbb{R}$.

In 2009, Rassias and Kim [31] investigated the Hyers-Ulam stability of Cauchy and Jensen type additive mappings in quasi-β-normed spaces with the following definition of a quasi-β-norm:

Definition 1.2. Let β be a real number with $0 < \beta \leq 1$ and $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let X be a linear space over a field $\mathbb{K}$. A *quasi-β-norm* $||\cdot||$ is a real-valued function on X satisfying the following properties:

1. $||x|| \geq 0$ for all $x \in X$ and $||x|| = 0$ if and only if $x = 0$
2. $||\lambda x|| = |\lambda|^\beta ||x||$ for all $\lambda \in \mathbb{K}$ and all $x \in X$
3. There is a constant $K \geq 1$ such that $||x + y|| \leq K(||x|| + ||y||)$ for all $x, y \in X$.

The pair $(X, ||\cdot||)$ is called a *quasi-β-normed space* if $||\cdot||$ is a quasi-β-norm on X. A smallest possible constant K is called the modulus of concavity of $||\cdot||$. A quasi-β-Banach space is a complete quasi-β-normed space. A quasi-β-norm $||\cdot||$ is called a (β, p)-norm $(0 < p \leq 1)$ if the property (3) of the Definition 1.2 takes the form $||x+y||^p \leq ||x||^p + ||y||^p$ for all $x, y \in X$. In this case, a quasi-β-Banach space is referred to as a (β, p)-Banach space; see [4, 31, 32].
Combining this norm and the fuzzy anti-norm on a linear space X with its triangle inequality property for all $x, y \in X$ and $t \in \mathbb{R}$ we, then, define a new fuzzy normed space as follows.

Definition 1.3. Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0, 1]$ is called a *quasi-fuzzy anti-β-norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

1. $N(x, t) = 1$ for $t \leq 0$
2. $N(x, t) = 0$ if and only if $x = 0$ for all $t > 0$
3. $N(cx, t) = N\left(x, \dfrac{t}{|c|^\beta}\right)$ for $c \neq 0$ and $0 < \beta \leq 1$
4. There is a constant $K \geq 1$ such that $N(x + y, t) \leq K(N(x, t) + N(y, t))$ for all $x, y \in X$ and $t \in \mathbb{R}$

5. $N(x,t)$ is a non-increasing function of $t \in \mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 0$,
6. for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is said to be a *quasi-fuzzy anti-β-normed linear space.*

EXAMPLE 1.4. *Let $(X, ||\cdot||)$ be a β-normed space. We use this normed space to define*

$$N(x,t) = \begin{cases} \dfrac{||x||}{t + ||x||} & \text{if} \quad t > 0, x \in \mathbb{R} \\ 1 & \text{otherwise} \end{cases}$$

where $x \in X$. Then we note that

$$N(cx,t) = \frac{||cx||}{t + ||cx||} = \frac{||x||}{\frac{t}{|c|^\beta} + ||x||} = N(x, \frac{t}{|c|^\beta})$$

for all $x \in X$ and $c \in \mathbb{R}$ $(c \neq 0, 0 < \beta \leq 1)$ and conclude that (X, N) a quasi-fuzzy anti-β-normed space induced by the β-norm $|| \cdot ||$.

Noting that the every fuzzy metric space in the sense of Kramosil and Michalek, and George and Veeramani is metrizable i.e., the induced topology can be induced by a usual metric (see e.g. S. Romaguera and V.Gregori [15]) we have an equivalence of topological type from the quasi-fuzzy anti-β-norm we just defined as one from the anti-fuzzy norm (Definition 1.1). However, our focus in the article is to study the *stability* of a "new type" of cubic functional equations (1.5) as we presented earlier on this quasi-β-normed space. Lastly in the long introduction we define the convergence of a sequence $\{x_n\}$ in a quasi-fuzzy anti-β-normed space X as follows.

DEFINITION 1.5. Let (X, N) be a quasi-fuzzy anti-β-normed space. A sequence $\{x_n\}$ in X is said to be convergent or converge if there is an element $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 0$ for all $t > 0$. If the sequence $\{x_n\}$ converges to x then x is called the limit of the sequence $\{x_n\}$ denoted by $N - \lim_{x\to\infty} x_n = x$.

In this paper, using both the *direct method* and the *fixed point method* we prove the generalized Hyers-Ulam stability of the generalized cubic functional equation (1.5) in a quasi-fuzzy anti-β-normed linear space we just defined above (Definition 1.3). In Section 2 we establish the general solution of the cubic functional equation (1.5) applying the symmetric n-additive mappings for the cubic functional equation (1.5) that will be explained in detail in the Section. In Section 3 we prove the generalized

Hyers-Ulam stability of the cubic functional equation (1.5) in a quasi-fuzzy anti-β-normed linear space X by using the direct method. Finally, we obtain, in Section 4, the generalized Hyers-Ulam stability of the generalized cubic functional equation (1.5) with the fixed point theorem of the alternative.

2. The General Solution

In this section we study the general solution of the cubic functional equation (1.5) by introducing and applying n-additive symmetric mappings and their properties that are found in [35, 40]. Before we proceed, let us give some basic backgrounds of n-additive symmetric mappings. Let X and Y be real vector spaces and n a positive integer. A function $A_n : X^n \to Y$ is called *n-additive* if it is additive in each of its variables. A function $A_n : X^n \to Y$ is said to be *symmetric* if $A_n(x_1, x_2, \cdots, x_n) = A_n(x_{\sigma(1)}, x_{\sigma(2)}, \cdots, x_{\sigma(n)})$ for every permutation $\{\sigma(1), \sigma(2), \cdots, \sigma(n)\}$ of $\{1, 2, \cdots, n\}$. If $A_n(x_1, x_2, \cdots, x_n)$ is an n-additive symmetric map, then $A^n(x)$ will denote the diagonal $A_n(x, x, \cdots, x)$ and $A^n(rx) = r^n A^n(x)$ for all $x \in X$ and $r \in \mathbb{Q}$. Such a function $A^n(x)$ will be called a *monomial function of degree* n assuming $A^n(x) \not\equiv 0$. Moreover, the resulting function after substituting $x_1 = x_2 = \cdots = x_s = x$ and $x_{s+1}, x_{s+2}, \cdots = x_n = y$ in $A_n(x_1, x_2, \cdots, x_n)$ will be denoted by $A^{s,n-s}(x, y)$.

THEOREM 2.1. *A function $f : X \to Y$ is a solution of the functional equation (1.5) if and only if f is of the form $f(x) = A^3(x)$ for all $x \in X$, where $A^3(x)$ is the diagonal of the 3-additive symmetric mapping $A_3 : X^3 \to Y$.*

Proof. Assume that f satisfies the functional equation (1.5). Taking $x = y = 0$ in the equation (1.5) it's not hard to have $f(0) = 0$. Substituting $y = 0$ in (1.5) also gives

$$f(ax) - f(x) + a(a-1)f(x) - (a-1)(a+1)^2 f(x) = 0,$$

that is,

$$f(ax) = a^3 f(x) \tag{2.1}$$

for all $x \in X$. Similarly, when $x = 0$ in the equation (1.5) we have

$$(1 - a^3 + (a-1)(a+1)^2)f(y) + a(a-1)f(-y) = 0,$$

i.e.,

$$a(a-1)(f(y) + f(-y)) \tag{2.2}$$

for all $y \in X$. This observation leads us to $f(-y) = -f(y)$ for all $y \in X$ and hence f is an odd function. Rewriting the equation (1.5) as

$$\begin{aligned} f(x) &- \frac{1}{(a-1)(a+1)^2} f(ax+y) + \frac{1}{(a-1)(a+1)^2} f(x+ay) \\ &- \frac{a}{(a+1)^2} f(x-y) - \frac{1}{(a+1)^2} f(y) = 0 \end{aligned} \tag{2.3}$$

and applying Theorems 3.5 and 3.6 in [40] we express f as

$$f(x) = A^3(x) + A^2(x) + A^1(x) + A^0 \tag{2.4}$$

where A^0 is an arbitrary element in Y and $A^i(x)$ is the diagonal of the i-additive symmetric mapping $A_i : X^i \to Y$ for $i = 1, 2, 3$. Since f is odd and $f(0) = 0$ it follows that

$$f(x) = A^3(x) + A^1(x)$$

for all $x \in X$. By the property (2.1) of f and $A^n(rx) = r^n A^n(x)$ for all $x \in X$ and $r \in \mathbb{Q}$ we should obtain $A^1(x) = 0$ for all $x \in X$. Therefore we conclude that $f(x) = A^3(x)$ for all $x \in X$.
Conversely, let us assume that $f(x) = A^3(x)$ for all $x \in X$, where $A^3(x)$ is the diagonal of a 3-additive symmetric mapping $A_3 : X^3 \to Y$. Noting that

$$A^3(qx + ry) = q^3 A^3(x) + 3q^2 r A^{2,1}(x, y) + 3qr^2 A^{1,2}(x, y) + r^3 A^3(y)$$

and calculating simple computation for the equation (1.5) in term of $A^3(x)$, we show that the function f satisfies the cubic equation (1.5), which completes the proof. □

3. Generalized Hyers-Ulam Stability in a quasi-fuzzy anti-β-normed space

In this section we shall study the generalized Hyers-Ulam stability for the cubic functional equation (1.5) in a quasi-fuzzy anti-β-normed space. Throughout this section let Y be a quasi-fuzzy anti-β-Banach space with the norm $N(x, t)$, X a real linear space, and the constant K the modulus of concavity of the norm $N(x, t)$ unless otherwise stated. For a given function $f : X \to Y$ and all fixed integer $a (a \neq 0, \pm 1)$ let

$$\begin{aligned} D_a f(x, y) :=& f(ax + y) - f(x + ay) + a(a-1) f(x - y) \\ &- (a-1)(a+1)^2 f(x) + (a-1)^2 f(y) \end{aligned} \tag{3.1}$$

Theorem 3.1. *Suppose that there is a mapping $\psi : X^2 \times \mathbb{R} \to [0, \infty)$ for which a function $f : X \to Y$ satisfies $f(0) = 0$ and*

$$N(D_a f(x, y), t) \leq \psi(x, y, t) \tag{3.2}$$

and the series

$$\sum_{j=1}^{\infty} K^j \psi\left(\frac{x}{a^j}, \frac{y}{a^j}, \frac{t}{|a|^{3\beta j}}\right)$$

converges for all $x, y \in X$ and all $t > 0$. Then there exists a unique cubic function $C : X \to Y$ satisfying both the equation (1.5) and the following inequality

$$N(f(x) - C(x), t) \leq \sum_{j=1}^{\infty} K^j \psi\left(\frac{x}{a^j}, 0, \frac{t}{|a|^{3\beta(j-1)}}\right) \tag{3.3}$$

for all $x \in X$. Moreover, the function C is given by

$$C(x) = N - \lim_{n \to \infty} (a^3)^n f(\frac{x}{a^n}) \tag{3.4}$$

for all $x \in X$ and $t \in \mathbb{R}$.

Proof. Starting with $y = 0$ in (3.1) and using $f(0) = 0$ we have

$$N(f(ax) - a^3 f(x), t) \leq \psi(x, 0, t). \tag{3.5}$$

We note that replacing x with $\frac{x}{a}$ in the inequality (3.5) we obtain

$$N\left(f(x) - a^3 f\left(\frac{x}{a}\right), t\right) \leq \psi\left(\frac{x}{a}, 0, t\right) \tag{3.6}$$

for all $x \in X$ and all $t > 0$. Substituting $\frac{x}{a}$ for x in (3.6) again gives

$$N\left(f\left(\frac{x}{a}\right) - a^3 f\left(\frac{x}{a^2}\right), t\right) \leq \psi\left(\frac{x}{a^2}, 0, t\right) \tag{3.7}$$

and by the properties of the quasi-fuzzy anti-β norm we have

$$N\left(a^3 f\left(\frac{x}{a}\right) - a^6 f\left(\frac{x}{a^2}\right), |a|^{3\beta} t\right) \leq \psi\left(\frac{x}{a^2}, 0, t\right). \tag{3.8}$$

for all $x \in X$ and all $t > 0$. Then we scale the variable t by $t = \frac{t}{|a|^{3\beta}}$ to have

$$N\left(a^3 f\left(\frac{x}{a}\right) - a^6 f\left(\frac{x}{a^2}\right), t\right) \leq \psi\left(\frac{x}{a^2}, 0, \frac{t}{|a|^{3\beta}}\right). \tag{3.9}$$

Combining two inequalities (3.6) and (3.9) we have

$$
\begin{aligned}
N\left(f(x)-a^6 f\left(\frac{x}{a^2}\right),t\right) &\le K\left(\psi\left(\frac{x}{a},0,t\right)+\psi\left(\frac{x}{a^2},0,\frac{t}{|a|^{3\beta}}\right)\right) \\
&\le K\psi\left(\frac{x}{a},0,t\right)+K^2\psi\left(\frac{x}{a^2},0,\frac{t}{|a|^{3\beta}}\right).
\end{aligned}
\tag{3.10}
$$

for all $x \in X$ and all $t > 0$ since $K \ge 1$. By the mathematical induction with the constant $K \ge 1$ we have

$$
N\left(f(x)-(a^3)^n f\left(\frac{x}{a^n}\right),t\right) \le \sum_{j=1}^{n} K^j \psi\left(\frac{x}{a^j},0,\frac{t}{(|a|^{3\beta})^{j-1}}\right)
\tag{3.11}
$$

for all $x \in X$ and all $t > 0$ and also

$$
N\left((a^3)^m f\left(\frac{x}{a^m}\right)-(a^3)^n f\left(\frac{x}{a^n}\right),t\right) \le \sum_{j=n}^{m} K^j \psi\left(\frac{x}{a^j},0,\frac{t}{(|a|^{3\beta})^{j-1}}\right)
\tag{3.12}
$$

for all $x \in X$, all $t > 0$, and $m, n \in \mathbb{N}(n < m)$. Since the right-hand side of the previous inequality (3.12) is going to 0 as $n \to \infty$ the sequence $\{(a^3)^n f\left(\frac{x}{a^n}\right)\}$ is a Cauchy sequence in the quasi-fuzzy β-Banach space Y and hence we define

$$
C(x) = N - \lim_{n\to\infty} (a^3)^n f\left(\frac{x}{a^n}\right)
\tag{3.13}
$$

for all $x \in X$. Taking the limit on both sides in (3.11) as n tends to the infinity shows the inequality (3.3) as desired.
In order to prove that C satisfies the equation (1.5) substitute $\frac{x}{a^n}$, $\frac{y}{a^n}$, and $\frac{t}{|a|^{3\beta n}}$ for x, y, and t, respectively, in (3.2). Then we have

$$
N\left(D_a f\left(\frac{x}{a^n},\frac{y}{a^n}\right),\frac{t}{|a|^{3\beta n}}\right) \le \psi\left(\frac{x}{a^n},\frac{y}{a^n},\frac{t}{|a|^{3\beta n}}\right)
\tag{3.14}
$$

or equivalently

$$
N\left((a^3)^n D_a f\left(\frac{x}{a^n},\frac{y}{a^n}\right),t\right) \le \psi\left(\frac{x}{a^n},\frac{y}{a^n},\frac{t}{|a|^{3\beta n}}\right).
\tag{3.15}
$$

Taking the limit of the above inequality as $n \to \infty$ we show that C is a cubic function for all $x, y \in X$ and all $t > 0$.
Lastly, we need to prove the uniqueness of the cubic function C involving the function f. Let us assume that there is a cubic function $D : X \to Y$

satisfying (1.5) and the inequality (3.3). Then we do have

$$N(D(x)-C(x),t) = N\left(a^{3n}\left[D\left(\frac{x}{a^n}\right)-C\left(\frac{x}{a^n}\right)\right],t\right)$$
$$= N\left(a^{3n}\left[D\left(\frac{x}{a^n}\right)-f\left(\frac{x}{a^n}\right)+f\left(\frac{x}{a^n}\right)-C\left(\frac{x}{a^n}\right)\right],t\right)$$
$$\leq K\left(N\left(a^{3n}\left[D\left(\frac{x}{a^n}\right)-f\left(\frac{x}{a^n}\right)\right],t\right)+N\left(a^{3n}\left[f\left(\frac{x}{a^n}\right)-C\left(\frac{x}{a^n}\right)\right],t\right)\right)$$
$$= K\left(N\left(D\left(\frac{x}{a^n}\right)-f\left(\frac{x}{a^n}\right),\frac{t}{|a|^{3n\beta}}\right)+N\left(f\left(\frac{x}{a^n}\right)-C\left(\frac{x}{a^n}\right),\frac{t}{|a|^{3n\beta}}\right)\right)$$
$$\leq 2\frac{K}{K^n}\left(\sum_{j=n+1}^{\infty}K^j\psi\left(\frac{x}{a^j},0,\frac{t}{|a|^{3\beta(j-1)}}\right)\right)$$

for all $x,y\in X$ and all $t>0$. Taking the limit of the above inequality as $n\to\infty$ we show the uniqueness of C and it completes the proof of the theorem. □

By the symmetry of the structure of the generalized cubic functional equation (1.5) in scaling of the variable x and the function-value $f(x)$ as an approximation of the cubic function we immediately obtain the following consequence from Theorem 3.1.

Corollary 3.2. *Suppose that there is a mapping $\psi : X^2\times\mathbb{R}\to[0,\infty)$ for which a function $f : X\to Y$ satisfies $f(0)=0$ and*

$$N(D_af(x,y),t)\leq\psi(x,y,t) \tag{3.16}$$

and the series

$$\sum_{j=1}^{\infty}K^j\psi\left(a^jx,a^jb,|a|^{3\beta j}t\right)$$

converges for all $x,y\in X$ and $t\in\mathbb{R}$. Then there exists a unique cubic function $C : X\to Y$ satisfying both the equation (1.5) and the following inequality

$$N(f(x)-C(x),t)\leq\sum_{j=1}^{\infty}K^j\psi\left(a^jx,0,|a|^{3\beta(j+1)}t\right) \tag{3.17}$$

for all $x\in X$. Moreover, the function C is given by

$$C(x)=\lim_{n\to\infty}\frac{1}{a^{3n}}f(a^nx) \tag{3.18}$$

for all $x\in X$ and $t\in\mathbb{R}$.

Proof. If we substitute ax for x in the inequality (3.6) then the proof follows from that of Theorem 3.1. □

From Theorem 3.1 we have the following result concerning a possible Cauchy difference for the stability of the generalized cubic functional equation (1.5) in the quasi-fuzzy anti-β-normed space corresponding the Aoki type of Cauchy difference for the linear functional equation (see [1]).

Corollary 3.3. *Let $\epsilon \geq 0$, $p > 3$ be a real number and X a β-normed space with norm $||\cdot||$. Assume that $f : X \to Y$ is a function satisfying $f(0) = 0$ and*

$$N(D_a f(x,y),t) \leq \frac{\epsilon(||x||^p + ||y||^p)}{t} \tag{3.19}$$

for all $x, y \in X$ and all $t > 0$. Then $C(x) := N - \lim_{n\to\infty} a^{3n} f\left(\frac{x}{a^n}\right)$ exists for each $x \in X$and defines the generalized cubic function $C : X \to Y$ such that

$$N(f(x) - C(x),t) \leq \frac{\epsilon K}{|a|^{p\beta} - K|a|^{3\beta}} \frac{||x||^p}{t} \tag{3.20}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.1 by taking $\psi(x,y,t) = \dfrac{\epsilon(||x||^p + ||y||^p)}{t}$ for all $x, y \in X$ and $t > 0$. □

4. General Hyers-Ulam Stability in a Quasi-fuzzy β-Banach Space: A Fixed Point Theorem of the Alternative Approach

In this section we will investigate the Hyers-Ulam-Rassias stability of the cubic functional equation (1.5) which is introduced earlier in previous sections

$$\begin{aligned} &f(ax+y) - f(x+ay) + a(a-1)f(x-y) \\ &= (a-1)(a+1)^2 f(x) - (a-1)(a+1)^2 f(y) \end{aligned}$$

for all $x, y \in X$ by the approach of the fixed point of the alternative. In order to give our results in this section it is convenient to state the definition of a generalized metric on a set X and a result on a fixed point theorem of the alternative by Diaz and Margolis [12].
Let X be a set. A function $d : X \times X \to [0,\infty]$ is called a generalized metric on X if d satisfies

1. $d(x,y) = 0$ if and only if $x = y$;
2. $d(x,y) = d(y,x)$ for all $x, y \in X$;
3. $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Theorem 4.1. *(Diaz and Margolis [12]) Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each element $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integers n or there exists a positive n_0 such that*

1. *$d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;*
2. *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
3. *y^* is the unique fixed point of J in the set $Y = \{y \in X | d(J^{n_0}x, y) < \infty\}$;*
4. *$d(y, y^*) \leq (1/(1-L))d(y, Jy)$ for all $y \in Y$.*

As we used the notations in the previous sections we assume that X is a vector space and (Y, N) is a quasi-fuzzy β-Banach space in this section. A set $\mathbb{R}_+$ denotes the set of all nonnegative real numbers.

Theorem 4.2. *Suppose that a function $\phi : X^2 \to ([0, \infty))$ is given and there exists a constant L with $0 < L < 1$ such that*

$$\phi(x, y) \leq \frac{L}{|a|^{3\beta}}\phi(ax, ay) \tag{4.1}$$

for all $x, y \in X$. Furthermore, let $f : X \to Y$ be a mapping such that $f(0) = 0$ and

$$N(D_a f(x, y), t) \leq \frac{\phi(x, y)}{t + \phi(x, y)} \tag{4.2}$$

for all $x, y \in X$ and all $t > 0$.
Then there exists the unique generalized cubic function $C : X \to Y$ defined by $C(x) := N - \lim_{n\to\infty} a^{3n} f(\frac{x}{a^n})$ such that

$$N(f(x) - C(x), t) \leq \frac{L\phi(x, 0)}{|a|^{3\beta}(1-L)t + L\phi(x, 0)} \tag{4.3}$$

for all $x \in X$ and all $t > 0$.

Proof. First, we put $y = 0$ in the equation (1.5) to obtain

$$N(f(ax) - a^3 f(x), t) \leq \frac{\phi(x, 0)}{t + \phi(x, 0)} \tag{4.4}$$

for $x \in X$ and all $t > 0$. Substituting $x = \dfrac{x}{a}$ in the equation (4.4) just above we have

$$N\left(f(x) - a^3 f\left(\frac{x}{a}\right), t\right) \leq \frac{\phi\left(\frac{x}{a}, 0\right)}{t + \phi\left(\frac{x}{a}, 0\right)}. \tag{4.5}$$

Applying the assumption on $\phi(x,y)$ in (4.1) we express the inequality (4.5) as

$$N\left(f(x)-a^3 f\left(\frac{x}{a}\right),t\right) \le \frac{\frac{L}{|a|^{3\beta}}\phi(x,0)}{t+\frac{L}{|a|^{3\beta}}\phi(x,0)} \tag{4.6}$$

for all $x \in X$ and all $t > 0$. Letting $t = \dfrac{L}{|a|^{3\beta}}t$ we have

$$N\left(f(x)-a^3 f\left(\frac{x}{a}\right),\frac{L}{|a|^{3\beta}}t\right) \le \frac{\phi(x,0)}{t+\phi(x,0)} \tag{4.7}$$

for all $x \in X$ and all $t > 0$.
Now we define a set S as

$$S := \{g : X \to X \quad \text{with} \quad g(0) = 0\}$$

and then a mapping d on $S \times S$ by

$$d(g,h) = \inf\{\mu \in \mathbb{R}_+ : N(g(x)-h(x),\mu t) \le \frac{\phi(x,0)}{t+\phi(x,0)}, \forall x \in X, t > 0\}$$

where $\inf \emptyset = \infty$ as a definition. Then (S,d) is a complete generalized metric space; see Lemma 2.1 in [23]. Now we also define a linear mapping $J : S \to S$ by

$$Jg(x) := a^3 g\left(\frac{x}{a}\right)$$

for all $x \in X$. Let $g, h \in S$ be given such that $d(g,h) = \epsilon$. Then we have

$$N\left(g(x)-h(x),\epsilon t\right) \le \frac{\phi(x,0)}{t+\phi(x,0)}$$

and

$$\begin{aligned} N\left(Jg(x)-Jh(x),L\epsilon t\right) &= N\left(a^3 g\left(\frac{x}{a}\right)-a^3 h\left(\frac{x}{a}\right),L\epsilon t\right) \\ &= N\left(g\left(\frac{x}{a}\right)-h\left(\frac{x}{a}\right),\frac{L}{|a|^{3\beta}}\epsilon t\right) \le \frac{\phi\left(\frac{x}{a},0\right)}{\frac{L}{|a|^{3\beta}}t+\phi\left(\frac{x}{a},0\right)} \\ &\le \frac{\phi(x,0)}{t+\phi(x,0)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. Thus we have the fact that $d(g,h) = \epsilon$ implies $d(Jg,Jh) \le L\epsilon$ to obtain

$$d(Jg,Jh) \le Ld(g,h).$$

for all $g, h \in S$.
Now from the observation of the inequality (4.7) we have $d(f,Jf) \le$

$\frac{L}{|a|^{3\beta}}$. Therefore we may conclude that by the fixed point theorem of the alternative (Theorem 4.1) there exists a mapping $C : X \to Y$ such that

1. C is a fixed point of J, that is,

$$C\left(\frac{x}{a}\right) = \frac{1}{a^3}C(x) \tag{4.8}$$

   for all $x \in X$. In other words, The mapping C is a unique fixed point of J in the set $M = \{g \in S | d(f,g) < \infty\}$ satisfying the equation (4.8) such that there is a $\mu \in (0,\infty)$ such that

$$N(f(x) - C(x), \mu t) \le \frac{\phi(x,0)}{t + \phi(x,0)}$$

   for all $x \in X$ and all $t > 0$;
2. $d(J^n f, C) \to 0$ as $n \to \infty$, which implies the following limit

$$N - \lim_{n\to\infty} a^{3n} f\left(\frac{x}{a^n}\right) = C(x)$$

   for all $x \in X$.
3. $d(f,C) \le \left(\frac{1}{1-L}\right) d(f,Jf)$ and hence we have

$$d(f,C) \le \left(\frac{1}{1-L}\right) \cdot \left(\frac{L}{|a|^{3\beta}}\right) = \left(\frac{L}{|a|^{3\beta}(1-L)}\right)$$

   and

$$N\left(f(x) - C(x), \frac{L}{|a|^{3\beta}(1-L)}t\right) \le \frac{\phi(x,0)}{t + \phi(x,0)}$$

   for all $x \in X$ and all $t > 0$. Replacing t by $\frac{|a|^{3\beta}(1-L)}{L}t$ we have

$$N\left(f(x) - C(x), t\right) \le \frac{L\phi(x,0)}{|a|^{3\beta}(1-L)t + L\phi(x,0)}$$

   for all $x, y \in X$ and all $t > 0$, which proves the inequality (4.3).

Lastly, in order to prove that the mapping $C : X \to Y$ is the generalized cubic function we let $x = \frac{x}{a^n}$ and $y = \frac{y}{a^n}$ in the inequality (4.2). Then we have

$$N\left(a^{3n} D_a f\left(\frac{x}{a^n}, \frac{y}{a^n}\right), |a|^{3\beta n} t\right) \le \frac{\phi(\frac{x}{a^n}, \frac{y}{a^n})}{t + \phi(\frac{x}{a^n}, \frac{y}{a^n})}$$

for all $x, y \in X$, all $t > 0$, and all $n \in \mathbb{N}$. Scaling $t = \frac{t}{|a|^{3\beta n}}$ we, then, have

$$N\left(a^{3n} D_a f\left(\frac{x}{a^n}, \frac{y}{a^n}\right), t\right) \leq \frac{\phi(\frac{x}{a^n}, \frac{y}{a^n})}{\frac{t}{|a|^{3\beta n}} + \phi(\frac{x}{a^n}, \frac{y}{a^n})} \leq \frac{L^n \phi(x, y)}{t + L^n \phi(x, y)}$$

for all $x, y \in X$, all $t > 0$, and all $n \in \mathbb{N}$. Since the right hand side of the above, $\dfrac{L^n \phi(x, y)}{t + L^n \phi(x, y)}$, approaches zero as n tends to the infinity for all $x, y \in X$ and all $t > 0$, we prove

$$N(D_a C(x, y), t) = 0$$

for all $x, y \in X$ and all $t > 0$ as we claimed and it completes the proof. □

Let us give the Cauchy difference type stability of the generalized cubic functional equation (1.5) as a consequence of Theorem 4.2 as follows.

Corollary 4.3. *Let $\epsilon \geq 0$, $p > 3$ be a real number, and X a normed linear space with norm $||\cdot||$. Suppose $f : X \to Y$ is a function satisfying $f(0) = 0$ and*

$$N(D_a f(x, y), t) \leq \frac{\epsilon(||x||^p + ||y||^p)}{t + \epsilon(||x||^p + ||y||^p)} \tag{4.9}$$

for all $x, y \in X$ and all $t > 0$. Then $C(x) := N-\lim_{n\to\infty} a^{3n}\left(\dfrac{x}{a^n}\right)$ exists uniquely and defines a generalized cubic function $C : X \to Y$ such that

$$N(f(x) - C(x), t) \leq \frac{\epsilon |a|^{3\beta} ||x||^p}{|a|^{3\beta}(|a|^{p\beta} - |a|^{3\beta})t + \epsilon |a|^{3\beta} ||x||^p} \tag{4.10}$$

for all $x, y \in X$ and all $t > 0$.

Proof. This proof follows from Theorem 4.2 by taking $\phi(x, y) = \epsilon(||x||^p + ||y||^p)$ for all $x, y \in X$ and $L = |a|^{(3-p)\beta}$.

□

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