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PROJECTIONS OF ALGEBRAIC VARIETIES WITH ALMOST LINEAR PRESENTATION I

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Abstract. Let X be a reduced closed subscheme in $\mathbb{P}^n$ and

$$\pi_q : X \to Y = \pi_q(X) \subset \mathbb{P}^{n-1}$$

be an isomorphic projection from the center $q \in \mathbb{P}^n \setminus X$. Suppose that the minimal free presentation of I_X is of the following form

$$R(-3)^{\beta_{2,1}} \oplus R(-4) \to R(-2)^{\beta_{1,1}} \to I_X \to 0.$$

In this paper, we prove that $H^1(\mathcal{I}_X(k)) = H^1(\mathcal{I}_Y(k))$ for all $k \geq 3$. This implies that Y is k-normal if and only if X is k-normal for $k \geq 3$. Moreover, we also prove that $\operatorname{reg}(Y) \leq \max\{\operatorname{reg}(X), 4\}$ and that I_Y is generated by homogeneous polynomials of degree ≤ 4.

1. Introduction

Let V be a vector space of dimension $n+1$ over an algebraically closed field k with a basis $x_0, \dots, x_n$. If $X \subset \mathbb{P}^n = \mathbb{P}(V)$ is a nondegenerate reduced closed subscheme, we write I_X for the defining saturated ideal of X in the polynomial ring $R = \operatorname{Sym}(V) = k[x_0, \dots, x_n]$.

We say that X satisfies property $\mathbf{N}_{d,p}$, for some $d \geq 2$, if the ideal I_X is generated in degrees $\leq d$ and

$$\beta^R_{i,j} = \dim_k \operatorname{Tor}^R_i(R/I_X, k)_{i+j} = 0 \text{ for all } j \geq d \text{ and for all } i \leq p.$$

If $d = 2$ and $p \geq 2$ then the minimal free resolution of R/I_X is of the following form

$$\cdots \to R(-3)^{\beta^R_{2,1}} \to R(-2)^{\beta^R_{1,1}} \to R \to R/I_X \to 0.$$

There has been a great deal of research on this condition (cf. [1, 2, 3, 4, 5, 7, 8]). In particular, the authors in [1] have proved that if

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$\pi_q : X \to Y = \pi_q(X) \subset \mathbb{P}^{n-1}$ is an isomorphic projection and X satisfies $\mathbf{N}_{2,p}$, $p \geq 2$ then

- $H^1(\mathcal{I}_X(k)) = H^1(\mathcal{I}_Y(k)) \quad$ for all $k \geq 2$;
- Y satisfies $\mathbf{N}_{3,p-1}$. Hence the ideal I_Y is generated in degree ≤ 3.
- $\operatorname{reg}(Y) \leq \max\{\operatorname{reg}(X), 3\}$.

In this paper, we slightly generalize these results to the case that I_X has an almost linear presentation, i.e., the minimal free resolution of R/I_X is of the following form:

$$\cdots \to R(-3)^{\beta^R_{2,1}} \oplus R(-4) \to R(-2)^{\beta^R_{1,1}} \to R \to R/I_X \to 0.$$

In this case, we will show that $H^1(\mathcal{I}_X(k)) = H^1(\mathcal{I}_Y(k))$ for all $k \geq 3$, which implies that Y is k-normal if and only if X is k-normal for every $k \geq 3$. Moreover, we also prove that $\operatorname{reg}(Y) \leq \max\{\operatorname{reg}(X), 4\}$ and I_Y is generated in degree ≤ 4.

Partial elimination ideals introduced by M. Green ([6]) and the elimination mapping cone theorem ([1]) will be used to prove our results. In particular, the regularity of the first partial elimination ideal $K_1(I_X)$ will play a critical role in the proof of our result.

2. Mapping Cone Construction and Partial Elimination Ideals

Let X be a reduced, nondegenerate closed subscheme in $\mathbb{P}^n$ and let $\pi_q : X \to Y = \pi_q(X) \subset \mathbb{P}^{n-1}$ be a projection from the center $q \in \mathbb{P}^n \setminus X$. Let $S = k[x_1, \ldots, x_n]$ and $R = k[x_0, \ldots, x_n]$ be the polynomial rings which are coordinate rings of $\mathbb{P}^{n-1}$ and $\mathbb{P}^n$ respectively. Note that the coordinate ring R/I_X of X can be considered as a graded S-module by the inclusion map $0 \to S \to R$. We write

$$\beta^S_{i,j} = \dim_k \operatorname{Tor}^S_i(R/I_X, k)_{i+j}$$

for the Betti numbers of R/I_X as a graded S-module.

2.1. Mapping cone construction

The mapping cone under projection and its related long exact sequence is our starting point to understand algebraic and geometric structures of projections.

Consider the graded S-module map $\varphi : R/I_X(-1) \xrightarrow{\times x_0} R/I_X$. Then we have the map $\overline{\varphi}$ on the graded Koszul complex of R/I_X over S, which induces the following long exact sequence by the mapping cone construction:

Theorem 2.1 (Theorem 3.2 in [1]). *For a graded R-module M, we have the following long exact sequence:*

$$\longrightarrow \mathrm{Tor}_i^S(M,k)_{i+j} \longrightarrow \mathrm{Tor}_i^R(M,k)_{i+j} \longrightarrow \mathrm{Tor}_{i-1}^S(M,k)_{i+j-1} \longrightarrow$$

$$\stackrel{\delta}{\to} \mathrm{Tor}_{i-1}^S(M,k)_{i+j} \longrightarrow \mathrm{Tor}_{i-1}^R(M,k)_{i+j} \longrightarrow \mathrm{Tor}_{i-2}^S(M,k)_{i+j-1} \stackrel{\delta}{\to} \cdots$$

whose connecting homomorphism δ is the multiplicative map $\times x_0$.

The following proposition can be proved by Theorem 2.1 ([2, Proposition 2.3]).

Proposition 2.2. *Let X be a reduced, nondegenerate closed subscheme in $\mathbb{P}^n$ and let $\pi_q : X \to Y = \pi_q(X) \subset \mathbb{P}^{n-1}$ be the projection from the center $q \in \mathbb{P}^n \setminus X$. Suppose that X satisfies property $\mathbf{N}^R_{d,p}$ for $d \geq 2$ and $p \geq 2$. Then we have*

(a) *R/I_X satisfies property $\mathbf{N}^S_{d,p-1}$ as a finitely generated graded S-module.*
(b) *$\beta^S_{i-1,d-1} \leq \beta^R_{i,d-1}$ for each i with $1 \leq i \leq p$.*

2.2. Partial elimination ideals

Let X be a reduced closed subscheme in $\mathbb{P}^n$ and let

$$\pi_q : X \to Y = \pi_q(X) \subset \mathbb{P}^{n-1}$$

be the isomorphic projection from the center $q = [1 : 0 : \cdots : 0] \in \mathbb{P}^n$.

For the degree lexicographic order, if $f \in I_X$ has leading term $\mathrm{in}(f) = x_0^{d_0} \cdots x_n^{d_n}$, we set $d_0(f) = d_0$, the leading power of x_0 in f. Then partial elimination ideals of I_X are defined as follows, which was given by M. Green in [6].

Definition 2.3 ([6]). Let $I_X \subset R$ be the defining ideal of X and let

$$\tilde{K}_i(I_X) = \bigoplus_{m \geq 0} \{f \in (I_X)_m \mid d_0(f) \leq i\}.$$

If $f \in \tilde{K}_i(I_X)$, we may write uniquely $f = x_0^i g + h$ where $g \in S$ and $h \in R$ with $d_0(h) < i$. Now we define $K_i(I_X)$ by the image of $\tilde{K}_i(I_X)$ in S under the map $f \mapsto g$ and we call $K_i(I_X)$ the i-th partial elimination ideal of I_X. Note that $\tilde{K}_i(I_X)$ and $K_i(I_X)$ are S-modules.

Remark 2.4. If $f = x_0 g + h \in \tilde{K}_1(I_X)_d$ then $g \in S_{d-1}$ and $h \in S_d$. Hence we have $(g,h) \in S(-1) \oplus S$.

The following proposition shows the geometric meaning of partial elimination ideals.

Proposition 2.5. [6] *Suppose that $\pi_q : X \to Y = \pi_q(X) \subset \mathbb{P}^{n-1}$ be a projection from the center $q = [1 : 0 : \cdots : 0] \in \mathbb{P}^n$. Then, set theoretically, the i-th partial elimination ideal $K_i(I_X)$ is the defining ideal of $Z_i = \{q \in Y \mid \mathrm{mult}_q(Y) \geq i+1\}$ for every $i \geq 0$. In particular, the defining ideal of Y is $K_0(I_X) = I_X \cap S$.*

Let $\pi_q : X \to Y \subset \mathbb{P}^{n-1}$ be a projection from the center $q = [1 : 0 \cdots : 0] \in \mathbb{P}^n \setminus X$. Suppose that X satisfies $\mathbf{N}_{2,1}$, i.e., I_X is generated in degree 2. Then there should be a quadratic polynomial $F \in I_X$ such that

$$F = x_0^2 + x_0 h_1 + h_0 \in I_X, \text{ where } h_1 \in (S)_1 \text{ and } h_2 \in (S)_2.$$

Thus we see that $K_2(I_X) = (1) = S$, which implies that $K_i(I_X) = S$ for all $i \geq 2$.

Let $G \in (R)_d$ be a homogeneous polynomial of degree d. Using the fact that $x_0^2 \equiv x_0 h_1 + h_0 \pmod{I_X}$, we have that

$$G \equiv x_0 g_1 + g_0 \mod I_X. \tag{2.1}$$

for some polynomials $g_1 \in (S)_{d-1}$, $g_0 \in (S)_d$. Hence we have a S-module map

$$\varphi_0 : S(-1) \oplus S \to R/I_X \to 0$$

defined by $\varphi_0(g_1, g_2) = [x_0 g_1 + g_0]$. Note that $(g_1, g_2) \in \ker(\varphi_0)$ if and only if $x_0 g_1 + g_0 \in I_X$. So we see that $\ker(\varphi_0) \cong \tilde{K_1}(I_X)$. Consequently, we have the following lemma.

Lemma 2.6. *Let X be a reduced closed subscheme in $\mathbb{P}^n$ and $\pi_q : X \to Y = \pi_q(X) \subset \mathbb{P}^{n-1}$ be the isomorphic projection from the center $q = [1 : 0 \cdots : 0] \in \mathbb{P}^n \setminus X$. Suppose that X satisfies $\mathbf{N}_{2,1}$. Then we have*

(a) *$K_i(I_X) = S$ for all $i \geq 2$.*
(b) *$0 \to \tilde{K_1}(I_X) \to S(-1) \oplus S \to R/I_X \to 0$.*

3. Main result

Theorem 3.1. *Let X be a reduced, nondegenerate closed subscheme in $\mathbb{P}^n$ and $\pi_q : X \to Y = \pi_q(X) \subset \mathbb{P}^{n-1}$ be an isomorphic projection from the center $q \in \mathbb{P}^n \setminus X$. Suppose that the minimal free resolution of R/I_X is of the following form*

$$R(-3)^{\beta_{2,1}} \oplus R(-4) \to R(-2)^{\beta_{1,1}} \to R \to R/I_X \to 0. \tag{3.1}$$

Then we have

(a) *$H^1(\mathcal{I}_X(k)) = H^1(\mathcal{I}_Y(k))$ for all $k \geq 3$.*

(b) $\mathrm{reg}(Y) \leq \min\{\mathrm{reg}(X), 4\}$.
(c) *the ideal* I_Y *is generated by polynomials of degree* ≤ 4.

Proof. We may assume that $q = [1 : 0 : \cdots : 0] \in \mathbb{P}^n \setminus X$ by a change of coordinates. For an isomorphic projection $\pi_q : X \to Y \subset \mathbb{P}^{n-1}$, we have a natural map:

$$\tilde{\alpha}_d : (S/I_Y)_d \to (R/I_X)_d.$$

Note that $\mathrm{coker}(\tilde{\alpha}_d)$ is vanishing for sufficiently large $d > 0$. First let us show that if R/I_X has the minimal free resolution of the form in (3.1) then $\tilde{\alpha}_d$ is an isomorphism for all $d \geq 3$. Since we have assumed that $\pi_q : X \to Y \subset \mathbb{P}^{n-1}$ is an isomorphic projection, the partial elimination ideals $K_i(I_X)$ are artinian ideals for all $i \geq 1$. Since I_X is generated in degree 2, we see from Lemma 2.6 (a) that $K_i(I_X) = S$ for all $i \geq 2$.

Now we claim that $K_1(I_X)_d = S_d$ for all $d \geq 2$. Indeed, if $f \in \tilde{K}_1(I_X)$, then we have

$$f = x_0 g_1 + g_0 \in I_X \quad \text{for some } (g_1, g_0) \in S(-1) \oplus S.$$

If we consider the map $\tilde{K}_1(I_X) \to K_1(I_X)(-1) \to 0$ defined by $f \mapsto g_1$ then we have the following exact sequence

$$0 \to I_Y \to \tilde{K}_1(I_X) \to K_1(I_X)(-1) \to 0. \tag{3.2}$$

Since we have

$$R(-3)^{\beta^R_{2,1}} \oplus R(-4) \to R(-2)^{\beta^R_{1,1}} \to R \to R/I_X \to 0,$$

it follows from Proposition 2.2 that the minimal free resolution of R/I_X as a graded S-module is of the form

$$\cdots \to S(-2)^{\beta^S_{1,1}} \oplus S(-3)^{\beta^S_{1,2}} \to S \oplus S(-1) \xrightarrow{\varphi_0} R/I_X \to 0. \tag{3.3}$$

Note that Lemma 2.6 (b) shows that $\tilde{K}_1(I_X)$ is the first syzygy module of R/I_X as a graded S-module. So we can consider the following diagram:

$$\begin{array}{cccc} S(-2)^{\beta^S_{1,1}} \oplus S(-3)^{\beta^S_{1,2}} & \longrightarrow & \tilde{K}_1(I_X) & \longrightarrow 0 \\ & \searrow & \downarrow & \\ & & K_1(I_X)(-1) & \\ & & \downarrow & \\ & & 0 & \end{array} \tag{3.4}$$

This shows that $K_1(I_X)$ is generated by linear forms and $\beta^S_{1,2}$ quadric forms. By Propostition 2.2 (b), the number of quadrics $\beta^S_{1,2}$ is at most

$\beta^R_{2,2} = 1$. This implies that $K_1(I_X)$ is an atinian ideal of complete intersection. Hence

$$K_1(I_X)_d = (S)_d \quad \text{for all } d \geq 2.$$

Now consider the following commutative diagram:
(3.5)

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & S & \to & S/I_Y & \to & 0 \\ & & \downarrow & & \downarrow \tilde{\alpha} & & \\ S(-2)^{\oplus \beta^S_{1,1}} \oplus S(-3)^{\oplus \beta^S_{1,2}} & \to & S(-1) \oplus S & \stackrel{\varphi_0}{\to} & R/I_X & \to & 0 \\ \| & & \downarrow & & \downarrow & & \\ S(-2)^{\oplus \beta^S_{1,1}} \oplus S(-3)^{\oplus \beta^S_{1,2}} & \stackrel{\mu}{\to} & S(-1) & \to & \mathrm{coker}(\tilde{\alpha}) & \to & 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Since $\operatorname{im} \mu = K_1(I_X)(-1) \subset S(-1)$ and $\operatorname{reg}(\operatorname{im} \mu) \leq 3$, we have that

$$\operatorname{reg}(\operatorname{coker} \tilde{\alpha}) = \operatorname{reg}(\operatorname{im} \mu) - 1 \leq 2, \tag{3.6}$$

which means that $(\operatorname{coker} \tilde{\alpha})_d = 0$ for all $d \geq 3$ or, equivalently, that

$$\tilde{\alpha}_d : (S/I_Y)_d \to (R/I_X)_d$$

is an isomorphism for each $d \geq 3$. Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & (S/I_Y)_d & \to & H^0(\mathcal{O}_Y(d)) & \to & H^1(\mathcal{I}_Y(d)) & \to & 0 \\ & & \downarrow \tilde{\alpha}_d & & \| & & \downarrow & & \\ 0 & \to & (R/I_X)_d & \to & H^0(\mathcal{O}_X(d)) & \to & H^1(\mathcal{I}_X(d)) & \to & 0, \end{array}$$

Then we see from the snake lemma that

$$H^1(\mathcal{I}_X(k)) = H^1(\mathcal{I}_Y(k)) \text{ for all } k \geq 3.$$

Consequently, Y is k-normal if and only if X is k-normal for all $k \geq 3$, which completes the proof (a).

On the other hand, recall that

$$0 \to (S/I_Y)_d \stackrel{\tilde{\alpha}}{\to} (R/I_X)_d$$

is isomorphism for all $d \geq 3$ and thus $\mathrm{reg}(\mathrm{coker}\,\tilde{\alpha}) \leq 2$. Then, from the right most column of the exact sequence in diagram (3.5), we have

$$\mathrm{reg}(S/I_Y) \leq \max\{\mathrm{reg}(R/I_X), \mathrm{reg}(\mathrm{coker}\,\tilde{\alpha}) + 1\},$$

which completes the proof of (b).

Finally, let us prove (c). For $d > 0$, we consider the following natural map

$$\cdots \to \mathrm{Tor}_1^S(K_1(I_X), k)_{d-1} \to \mathrm{Tor}_0^S(I_Y, k)_d \to \mathrm{Tor}_0^S(\tilde{K}_1(I_X), k)_d.$$

from the short exact sequence (3.2). Since $\tilde{K}_1(I_X)$ is generated in degree 3 (see the diagram (3.4)) and $K_1(I_X)$ is 2-regular, we conclude that $\mathrm{Tor}_0^S(I_Y, k)_d$ is zero for $d \geq 5$. This shows that I_Y is generated in degree ≤ 4, as we wished. □

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