

ON THE FUZZY STABILITY PROBLEM OF A QUADRATIC MAPPING WITH INVOLUTION

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ABSTRACT. We prove the generalized Hyers-Ulam-Rassias stability problem of the quadratic functional equation with involution in the fuzzy quasi β -normed space by using the fixed point method.

1. Introduction

The concept of stability problem of a functional equation was first posed by Ulam [23] concerning the stability of group homomorphisms; *Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \rightarrow G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \rightarrow G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?* Hyers [9] gave us a partial answer to the question of Ulam. His theorem was generalized in various directions. The very first author who generalized Hyers' theorem to the case of unbounded control functions was Aoki [1]. Also, Rassias [20] succeeded in extending the result of Hyers' theorem by weakening the condition for the Cauchy difference. Rassias' paper [20] has provided a lot of influence in the development of Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. In 1996, Isac and Rassias [10] were first to provide applications of new fixed point theorems for the proof of stability theory of functional equations. By using fixed point methods the stability problems of several functional equations have been extensively investigated by a number of authors; see [5], [6], [18] and [19].

Received Jun 9, 2018; Accepted Jul 11, 2018.

2010 Mathematics Subject Classification: Primary 39B52.

Key words and phrases: Hyers-Ulam-Rassias stability, quadratic mapping, involution, quasi- β -normed space, fixed point, fuzzy normed space.

The present research was conducted by the research fund of Dankook university in 2018.

Let X and Y be real vector spaces. If an additive function $\sigma : X \rightarrow Y$ satisfies $\sigma(\sigma(x)) = x$, for all $x \in X$, then we call it an *involution*; see [3] and [22]. The following functional equation

$$(1.1) \quad f(x+y) + f(x+\sigma(y)) = 2f(x) + 2f(y)$$

is called the quadratic functional equation with involution σ . Recently, Belaid et al [3]. proved the Hyers-Ulam-Rassias stability with involution in Banach space for this functional equation. Also, Jung and Lee [11] have proved the Hyers-Ulam-Rassias stability of the quadratic functional equation with involution in a complete β -normed space by using fixed point method.

In this paper we prove the generalized Hyers-Ulam-Rassias stability problem of the quadratic functional equation with involution(1.1) in the fuzzy quasi β -normed space by using the fixed point method.

2. Preliminaries

We will use the following definition to prove Hyers-Ulam-Rassias stability for the generalized quintic functional equation in the quasi β -normed space. Let β be a real number with $0 < \beta \leq 1$ and \mathbb{K} be either \mathbb{R} or \mathbb{C} .

DEFINITION 2.1. Let X be a linear space over a field \mathbb{K} . A *quasi β -norm* $\|\cdot\|$ is a real-valued function on X satisfying the following statements:

- (1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
- (2) $\|\lambda x\| = |\lambda|^\beta \cdot \|x\|$ for all $\lambda \in \mathbb{K}$ and all $x \in X$.
- (3) There is a constant $K \geq 1$ such that $\|x+y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a *quasi β -normed space* if $\|\cdot\|$ is a quasi β -norm on X . The smallest possible K is called the *modulus of concavity* of $\|\cdot\|$. A *quasi β -Banach space* is a complete quasi- β -normed space.

In 1984, Katsaras [12] and Wu and Fang [24] independently introduced a notion of a fuzzy norm. Since then some mathematicians have defined fuzzy metrics and norms on a linear space from various points of view; see [2], [8], [13], [25] and [16]. In 2003, Bag and Samanta [2] modified the definition of Cheng and Mordeson [7]. Bag and Samanta [2] introduced the following definition of fuzzy normed spaces. The notion of fuzzy stability of functional equations was given in the paper [17].

We will use the definition of fuzzy normed spaces to investigate a fuzzy version of Hyers-Ulam-Rassias stability in the fuzzy normed algebra setting.

DEFINITION 2.2. Let X be a real vector space. A function $N : X \times \mathbb{R} \rightarrow [0, 1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

- (N₁) $N(x, t) = 0$ for $t \leq 0$;
- (N₂) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;
- (N₃) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
- (N₄) $N(x + y, s + t) \geq \min \{N(x, s), N(y, t)\}$;
- (N₅) $N(x, \cdot)$ is a non-decreasing function of \mathbb{R} and $\lim_{t \rightarrow \infty} N(x, t) = 1$;
- (N₆) for $x \neq 0$, $N(x, \cdot)$ is continuous on \mathbb{R} .

The pair (X, N) is called a *fuzzy normed vector space*.

DEFINITION 2.3. Let X be a real vector space. A fuzzy norm $N : X \times \mathbb{R} \rightarrow [0, 1]$ is called a *quasi fuzzy β -norm* on X if (N₃) and (N₄) in Definition 2.2 are replaced by the following forms

$$(N'_3) \quad N(cx, t) = N(x, \frac{t}{|c|^\beta}) \quad (c \neq 0, 0 < \beta \leq 1).$$

and

$$(N'_4) \quad N(x+y, K(s+t)) \geq \min \{N(x, s), N(y, t)\} \quad (x, y \in X, s, t > 0),$$

respectively.

EXAMPLE 2.4. Let $(X, \|\cdot\|)$ be a real quasi β -normed space. Define

$$N(x, t) = \begin{cases} \frac{t}{t + \|x\|} & \text{when } t > 0, t \in \mathbb{R} \\ 0 & \text{when } t \leq 0, \end{cases}$$

where $x \in X$. Then (X, N) is a quasi fuzzy β -normed space.

Note that when $\beta = 1$, we call the quasi fuzzy β -norm a quasi fuzzy β -norm.

DEFINITION 2.5. Let (X, N) be a quasi fuzzy β -normed vector space. A sequence $\{x_n\}$ in X is said to be *convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n \rightarrow \infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $N\text{-}\lim_{n \rightarrow \infty} x_n = x$.

DEFINITION 2.6. Let (X, N) be a quasi fuzzy β -normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all integer $d > 0$, we have $N(x_{n+d} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a quasi fuzzy β -normed vector space is Cauchy. If each Cauchy sequence is convergent, then the quasi fuzzy β -normed space is said to be *quasi fuzzy complete* and the quasi fuzzy β -normed vector space is called a *quasi fuzzy Banach space*.

Now, we will state the theorem, the alternative of fixed point in a generalized metric space.

DEFINITION 2.7. Let X be a set. A function $d : X \times X \rightarrow [0, \infty]$ is called a *generalized metric* on X if d satisfies

- (1) $d(x, y) = 0$ if and only if $x = y$;
- (2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

THEOREM 2.8 (The alternative of fixed point [14], [21]). Suppose that we are given a complete generalized metric space (X, d) and a strictly contractive mapping $T : X \rightarrow X$ with Lipschitz constant $0 < L < 1$. Then for each given $x \in X$, either

$$d(T^n x, T^{n+1} x) = \infty \text{ for all } n \geq 0,$$

or there exists a natural number n_0 such that

1. $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$;
2. The sequence $\{T^n x\}$ is convergent to a fixed point y^* of T ;
3. y^* is the unique fixed point of T in the set

$$Y = \{y \in X \mid d(T^{n_0} x, y) < \infty\};$$

4. $d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in Y$.

3. Fuzzy fixed point stability over a Fuzzy Banach space

Let us fix some notations which will be used throughout this section. We assume X is a vector space and (Y, N) is a fuzzy Banach space. Using fixed point method, we will prove the Hyers-Ulam stability of the functional equation satisfying equation (3.1) in fuzzy Banach space. For a given mapping $f : X \rightarrow Y$ let

$$(3.1) \quad D_\sigma f(x, y) := f(x + y) + f(x + \sigma(y)) - 2f(x) - 2f(y)$$

for all $x, y \in X$, where $\sigma : X \rightarrow X$ is an involution.

THEOREM 3.1. Let β be a fixed real number with $0 < \beta \leq 1$ and let $\phi : X^2 \rightarrow [0, \infty)$ be a function such that there exists an $0 < L < 1$ with

$$(3.2) \quad \phi(2x, 2y) \leq 2^{2\beta} L \phi(x, y), \quad \phi(x + \sigma(x), y + \sigma(y)) \leq 2^{2\beta} L \phi(x, y)$$

for all $x, y \in X$. Let $f : X \rightarrow Y$ be a mapping satisfying

$$(3.3) \quad N(D_\sigma f(x, y), t) \geq \frac{t}{t + \phi(x, y)}$$

for all $x, y \in X$ and all $t > 0$. Then

$$Q(x) := N\text{-}\lim_{n \rightarrow \infty} \frac{1}{2^{2n}} \left[f(2^n x) + (2^n - 1)f(2^{n-1}(x + \sigma(x))) \right]$$

exists for each $x \in X$ and defines a quadratic mapping $Q : X \rightarrow Y$ such that

$$(3.4) \quad N(f(x) - Q(x), t) \geq \frac{2^{2\beta}(1-L)t}{2^{2\beta}(1-L)t + \phi(x, x)}$$

for all $x \in X$ and all $t > 0$.

Proof. First, let us define S to be the set of all functions $g : X \rightarrow Y$ and introduce a generalized metric on X as follows:

$$S := \{g : X \rightarrow X\}$$

and the mapping d defined on $S \times S$ by

$$d(g, h) = \inf\{\mu \in \mathbb{R}^+ \mid N(g(x) - h(x), \mu t) \geq \frac{t}{t + \phi(x, x)}, \forall x \in X \text{ and } t > 0\}$$

where $\inf \emptyset = +\infty$, as usual. Then (S, d) is a complete generalized metric space; see [15, Lemma 2.1]. For each $g, h \in X$, there exists a non-negative real number μ such that $d(g, h) \leq \mu$. We note that

$$N(g(x) - f(x), \mu t) \geq \frac{t}{t + \phi(x, x)}$$

for all $x \in X$ and $t > 0$.

By letting $y = x$ in the inequality (3.3), we have

$$(3.5) \quad N(f(2x) + f(x + \sigma(x)) - 2^2 f(x), t) \geq \frac{t}{t + \phi(x, x)}$$

for all $x \in X$ and all $t > 0$. Hence we may define an operator $T : S \rightarrow S$ by

$$T(g) = \frac{1}{2^2} [f(2x) + f(x + \sigma(x))]$$

for all $x \in X$.

$$\begin{aligned}
& N\left(T(g)(x) - T(h)(x), L\mu t\right) \\
&= N\left(\frac{1}{4}\left[g(2x) - h(2x) + g(x + \sigma(x)) - h(x + \sigma(x))\right], L\mu t\right) \\
&\geq \min \left\{ N\left(g(2x) - h(2x), |2|^{2\beta} L\mu t\right), \right. \\
&\quad \left. N\left(g(x + \sigma(x)) - h(x + \sigma(x)), |2|^{2\beta} L\mu t\right) \right\} \\
&\geq \frac{t}{t + \phi(x, x)}
\end{aligned}$$

for all $x \in X$ and $t > 0$. Hence we have $d(T(g), T(h)) \leq L\mu$. This implies that $d(T(g), T(h)) \leq L d(g, h)$, for $g, h \in S$. Thus T is strictly contractive because L is a constant with $0 < L < 1$. Next, the inequality (3.5) implies that

$$\begin{aligned}
N\left(T(f)(x) - f(x), t\right) &= N\left(f(2x) + f(x + \sigma(x)) - 2^2 f(x), |2|^{2\beta} t\right) \\
&\geq \frac{2^{2\beta} t}{2^{2\beta} t + \phi(x, x)}
\end{aligned}$$

or

$$(3.6) \quad N\left(T(f)(x) - f(x), t\right) \geq \frac{2^{2\beta} t}{2^{2\beta} t + \phi(x, x)}$$

for all $x \in X$ and $t > 0$. Replacing t by $\frac{1}{|2|^{2\beta}} t$ in the inequality (3.6), we have

$$N\left(T(f)(x) - f(x), \frac{1}{|2|^{2\beta}} t\right) \geq \frac{t}{t + \phi(x, x)}$$

for all $x \in X$ and $t > 0$. This means that $d(T(f), f) \leq \frac{1}{2^{2\beta}} < \infty$. Now, we claim that

$$(3.7) \quad T^n(f)(x) = \frac{1}{2^{2n}} \left[f(2^n x) + (2^n - 1) f(2^{n-1}(x + \sigma(x))) \right]$$

for all $x \in X$ and $n \in \mathbb{N}$. We denote that $T^0(f) = f$. The first step follows from the inequality (3.6), that is,

$$N\left(T(f)(x) - f(x), t\right) \geq \frac{t}{t + \frac{1}{2^{2\beta}} \phi(x, x)}.$$

For each $n \in \mathbb{N}$, we have

$$\begin{aligned}
 & N\left(T^n(f)(x) - T^{n-1}(f)(x), t\right) \\
 & \geq N\left(\frac{1}{2^{2n}}[f(2^n x) + (2^n - 1)f(2^{n-1}(x + \sigma(x)))] \right. \\
 & \quad \left. - \frac{1}{2^{n-1}}[f(2^{n-1}x) + (2^{n-1} - 1)f(2^{n-2}(x + \sigma(x)))] , t\right) \\
 & = N\left(\frac{1}{2^2}[f(2^n x) + f(2^{n-1}(x + \sigma(x)))] - f(2^{n-1}x) \right. \\
 & \quad \left. + \frac{2^{n-1} - 1}{2^2}[2f(2^{n-1}(x + \sigma(x))) - f(2^{n-2}(x + \sigma(x)))] , |2|^{2(n-1)\beta}t\right) \\
 & \geq \frac{2^{2(n-1)\beta}t}{2^{2(n-1)\beta}t + \frac{1}{2^{2\beta}}\phi(2^{n-1}x, 2^{n-1}x)} = \frac{t}{t + \frac{1}{2^{2\beta}}L^{n-1}\phi(x, x)}
 \end{aligned}$$

for all $x \in X$ and $t > 0$. Hence we have

$$N\left(T^n(f)(x) - T^{n-1}(f)(x), \frac{1}{|2|^{2\beta}}L^{n-1}t\right) \geq \frac{t}{t + \phi(x, x)}$$

for all $x \in X$ and $t > 0$. This implies that

$$(3.8) \quad d(T^n(f), T^{n-1}(f)) \leq \frac{1}{2^{2\beta}}L^{n-1} < \infty$$

as $n \rightarrow \infty$, where $0 < L < 1$. By the (2) of Theorem 2.8, there exists a mapping $Q : X \rightarrow Y$ which is a fixed point of T such that $d(T^n(f), Q) = 0$ as $n \rightarrow \infty$. Since $\lim_{n \rightarrow \infty} d(T^n(f), Q) = 0$, there exists a sequence $\{\mu_n\}$ in \mathbb{R} such that $\mu_n \rightarrow 0$ as $n \rightarrow \infty$ and $d(T^n f, Q) \leq \mu_n$ for $n \in \mathbb{N}$. The definition of d implies that

$$Q(x) := \text{N-} \lim_{n \rightarrow \infty} \frac{1}{2^{2n}} \left[f(2^n x) + (2^n - 1)f(2^{n-1}(x + \sigma(x))) \right]$$

for all $x \in X$. By the (4) of Theorem 2.8, we get $d(f, Q) \leq \frac{1}{1-L} d(T(f), f)$. Hence we have the inequality

$$d(f, Q) \leq \frac{1}{2^{2\beta}(1-L)}.$$

Thus the inequality (3.4) holds. For $x, y \in X$ and $t > 0$,

$$\begin{aligned}
& N\left(D_\sigma T^n(f)(x, y), t\right) \\
&= N\left([f(2^n(x+y)) + f(2^n(x+\sigma(y))) - 2f(2^n x) - 2f(2^n y)] \right. \\
&\quad \left. + (2^{n-1} - 1)[2f(2^{n-1}(x+y+\sigma(x+y))) - 2f(2^{n-1}(x+\sigma(x))) \right. \\
&\quad \left. - 2f(2^{n-1}(y+\sigma(y)))] , |2|^{2n\beta} t\right) \\
&\geq \min \left\{ N(D_\sigma f(2^n x, 2^n y), |2|^{2n\beta} t), \right. \\
&\quad \left. N(D_\sigma f(2^{n-1}(x+\sigma(x)), 2^{n-1}(y+\sigma(y)), |2|^{2n\beta} t) \right\} \\
&= \frac{2^{2n\beta} t}{2^{2n\beta} t + \phi(2^n x, 2^n y)} = \frac{t}{t + L^n \phi(x, y)} \rightarrow 1
\end{aligned}$$

as $n \rightarrow \infty$. Thus $Q(x) := \text{N-lim}_{n \rightarrow \infty} T^n(f)(x)$ is a quadratic mapping. The uniqueness of the quadratic mapping follows from (3) in Theorem 2.8. \square

COROLLARY 3.2. *Let $\theta \geq 0$, $p < 1$ be real numbers and let β be a real number with $\frac{p+1}{2} < \beta \leq 1$. Let X be a normed linear space with norm $\|\cdot\|$. Suppose $f : X \rightarrow Y$ is a mapping satisfying*

$$(3.9) \quad N(D_\sigma f(x, y), t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{(2^{2\beta} - 2^{p+1})t}{(2^{2\beta} - 2^{p+1})t + 2\theta\|x\|^p}$$

where $\|x + \sigma(x)\|^p \leq 2^{p+1}\|x\|^p$ for all $x \in X$ and $t > 0$.

Proof. Let $\phi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ and $L = 2^{p-2\beta+1}$. We note that $0 < L < 1$ and

$$\begin{aligned}
\phi(2x, 2x) &= 2^{p+1}\theta\|x\|^p = 2^{2\beta}L\phi(x, x) \\
\phi(x + \sigma(x), x + \sigma(x)) &= 2\theta\|x + \sigma(x)\|^p \leq 2 \cdot 2^{p+1}\theta\|x\|^p = 2^{2\beta}L\phi(x, x)
\end{aligned}$$

for all $x \in X$. The remains follow from the proof follows from Theorem 3.1. \square

COROLLARY 3.3. *Let $\theta \geq 0$, p and β be real numbers with $p < \beta \leq 1$. Let X be a normed linear space with norm $\|\cdot\|$. Suppose $f : X \rightarrow Y$ is*

a mapping satisfying

$$(3.10) \quad N(D_\sigma f(x, y), t) \geq \frac{t}{t + \theta(\|x\|^p \|y\|^p)}$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{(2^{2\beta} - 2^{2p})t}{(2^{2\beta} - 2^{2p})t + \theta\|x\|^{2p}}$$

where $\|x + \sigma(x)\|^p \leq 2^p\|x\|^p$ for all $x \in X$ and $t > 0$.

Proof. Let $\phi(x, y) = \theta(\|x\|^p \|y\|^p)$ for all $x, y \in X$ and $L = 2^{2(p-\beta)}$. Since $p < \beta \leq 1$, we know that $0 < L < 1$ and

$$\begin{aligned} \phi(2x, 2x) &= 2^{2p}\theta\|x\|^{2p} = 2^{2\beta}L\phi(x, x) \\ \phi(x + \sigma(x), x + \sigma(x)) &= \theta\|x + \sigma(x)\|^{2p} \leq 2^{2p}\theta\|x\|^{2p} = 2^{2\beta}L\phi(x, x) \end{aligned}$$

for all $x \in X$. The remains follow from the proof follows from Theorem 3.1. \square

THEOREM 3.4. Let β be a fixed real number with $0 < \beta \leq 1$ and let $\phi : X^2 \rightarrow [0, \infty)$ be a function such that there exists an $0 < L < 1$ with

$$(3.11) \quad \phi\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{L}{2^{2\beta}}\phi(x, y), \quad \phi(x + \sigma(x), y + \sigma(y)) \leq 2^\beta\phi(2x, 2y)$$

for all $x, y \in X$. Suppose that $f : X \rightarrow Y$ be a mapping satisfying

$$(3.12) \quad N(D_\sigma f(x, y), t) \geq \frac{t}{t + \phi(x, y)}$$

for all $x, y \in X$ and all $t > 0$. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$(3.13) \quad N(f(x) - Q(x), t) \geq \frac{2^{2\beta}(1 - L)t}{2^{2\beta}(1 - L)t + L\phi(x, x)}$$

for all $x \in X$ and all $t > 0$.

Proof. We will use the same definitions for the set S and the metric d as in the proof of Theorem 3.1. For each $g, h \in X$, there exists a non-negative real number μ such that $d(g, h) \leq \mu$. We note that

$$N(g(x) - f(x), \mu t) \geq \frac{t}{t + \phi(x, x)}$$

for all $x \in X$ and $t > 0$. To apply the fixed point method, we will define the contractive mapping $T : S \rightarrow S$ as in the proof of Theorem 2.8 and

then inductively define $T^n(f)(x)$. By letting $y = x$ in the inequality (3.12), we have

$$(3.14) \quad N\left(f(2x) + f(x + \sigma(x)) - 2^2 f(x), t\right) \geq \frac{t}{t + \phi(x, x)}$$

for all $x \in X$ and all $t > 0$. Replacing x by $\frac{x}{2}$ and $\frac{1}{4}(x + \sigma(x))$ in the inequality (3.14), respectively, we have

$$\begin{aligned} N\left(f(x) + f\left(\frac{1}{2}(x + \sigma(x))\right) - 2^2 f\left(\frac{x}{2}\right), t\right) &\geq \frac{t}{t + \phi\left(\frac{x}{2}, \frac{x}{2}\right)} \\ N\left(2f\left(\frac{1}{2}(x + \sigma(x))\right) - 2^2 f\left(\frac{1}{4}(x + \sigma(x))\right), t\right) \\ &\geq \frac{t}{t + \phi\left(\frac{1}{4}(x + \sigma(x)), \frac{1}{4}(x + \sigma(x))\right)} \end{aligned}$$

for all $x \in X$ and $t > 0$. By using the inequalities (3.11), we get

$$\begin{aligned} N\left(2^2 f\left(\frac{x}{2}\right) - f\left(\frac{1}{2}(x + \sigma(x))\right) - f(x), t\right) &\geq \frac{t}{t + \frac{L}{2^{2\beta}} \phi(x, x)} \\ N\left(f\left(\frac{1}{2}(x + \sigma(x))\right) - 2f\left(\frac{1}{4}(x + \sigma(x))\right), t\right) &\geq \frac{t}{t + \frac{L}{2^{2\beta}} \phi(x, x)} \end{aligned}$$

for all $x \in X$ and $t > 0$. Hence we may define an operator $T : S \rightarrow S$ by

$$T(g)(x) = 2^2 \left[f\left(\frac{x}{2}\right) - \frac{1}{2} f\left(\frac{1}{4}(x + \sigma(x))\right) \right]$$

for all $x \in X$. Now, for $g, h \in S$,

$$\begin{aligned} &N\left(T(g)(x) - T(h)(x), L\mu t\right) \\ &= N\left(2^2 \left[g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right) \right] + 2 \left[g\left(\frac{1}{4}(x + \sigma(x))\right) - h\left(\frac{1}{4}(x + \sigma(x))\right) \right], L\mu t\right) \\ &\geq \min \left\{ N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L\mu}{|2|^{2\beta}} t\right), \right. \\ &\quad \left. N\left(g\left(\frac{1}{4}(x + \sigma(x))\right) - h\left(\frac{1}{4}(x + \sigma(x))\right), \frac{L\mu}{|2|^\beta} t\right) \right\} \\ &\geq \frac{t}{t + \phi(x, x)} \end{aligned}$$

for all $x \in X$ and $t > 0$. Hence we have $d(T(g), T(h)) \leq L\mu$. This implies that $d(T(g), T(h)) \leq L d(g, h)$, for $g, h \in S$. Thus T is strictly

contractive because L is a constant with $0 < L < 1$. Also, we have

$$\begin{aligned}
 & N\left(T(f)(x) - f(x), t\right) \\
 &= N\left(2^2\left[f\left(\frac{x}{2}\right) - \frac{1}{2}f\left(\frac{1}{4}(x + \sigma(x))\right)\right] - f(x), L\mu t\right) \\
 &\geq \min \left\{ N\left(2^2f\left(\frac{x}{2}\right) - f\left(\frac{1}{2}(x + \sigma(x))\right) - f(x), t\right), \right. \\
 &\quad \left. N\left(f\left(\frac{1}{2}(x + \sigma(x))\right) - 2f\left(\frac{1}{4}(x + \sigma(x))\right), t\right) \right\} \\
 &\geq \frac{t}{t + \frac{L}{2^{2\beta}}\phi(x, x)}
 \end{aligned}$$

for all $x \in X$ and $t > 0$. That is,

$$(3.15) \quad N\left(T(f)(x) - f(x), t\right) \geq \frac{t}{t + \frac{L}{2^{2\beta}}\phi(x, x)}$$

for all $x \in X$ and $t > 0$. Replacing t by $\frac{L}{|2|^{2\beta}}t$ in the inequality (3.15), we have

$$N\left(T(f)(x) - f(x), \frac{L}{|2|^{2\beta}}t\right) \geq \frac{t}{t + \phi(x, x)}$$

for all $x \in X$ and $t > 0$. This means that $d(T(f), f) \leq \frac{L}{2^{2\beta}} < \infty$. Similar to the proof of Theorem 3.1, we have

$$(3.16) \quad T^n(f)(x) = 2^{2n}\left[f\left(\frac{1}{2^n}x\right) + \left(\frac{1}{2^n} - 1\right)f\left(\frac{1}{2^{n+1}}(x + \sigma(x))\right)\right]$$

for all $x \in X$ and $n \in \mathbb{N}$. We denote that $T^0(f) = f$. Also, we note that

$$N\left(T^n(f)(x) - T^{n-1}(f)(x), \frac{L^n}{|2|^{2\beta}}t\right) \geq \frac{t}{t + \phi(x, x)}$$

for all $x \in X$ and $t > 0$. This implies that

$$(3.17) \quad d(T^n(f), T^{n-1}(f)) \leq \frac{L^n}{2^{2\beta}} < \infty$$

as $n \rightarrow \infty$, where $0 < L < 1$. By the (2) of Theorem 2.8, there exists a mapping $Q : X \rightarrow Y$ which is a fixed point of T such that $d(T^n(f), Q) = 0$ as $n \rightarrow \infty$. Since $\lim_{n \rightarrow \infty} d(T^n(f), Q) = 0$, there exists a sequence $\{\mu_n\}$ in \mathbb{R} such that $\mu_n \rightarrow 0$ as $n \rightarrow \infty$ and $d(T^n f, Q) \leq \mu_n$ for $n \in \mathbb{N}$. The definition of d implies that

$$Q(x) := N\text{-}\lim_{n \rightarrow \infty} 2^{2n}\left[f\left(\frac{1}{2^n}x\right) + \left(\frac{1}{2^n} - 1\right)f\left(\frac{1}{2^{n+1}}(x + \sigma(x))\right)\right]$$

for all $x \in X$. By the (4) of Theorem 2.8, we get $d(f, Q) \leq \frac{1}{1-L} d(T(f), f)$. Hence we have the inequality

$$d(f, Q) \leq \frac{L}{2^{2\beta}(1-L)}.$$

Thus the inequality (3.13) holds. For $x, y \in X$ and $t > 0$,

$$\begin{aligned} & N\left(D_\sigma T^n(f)(x, y), t\right) \\ & \geq \min \left\{ N\left(D_\sigma f\left(\frac{1}{2^n}x, \frac{1}{2^n}y\right), \frac{1}{|2|^{2n\beta}}t\right), \right. \\ & \quad \left. N\left(D_\sigma f\left(\frac{1}{2^{n+1}}(x + \sigma(x)), \frac{1}{2^{n+1}}(y + \sigma(y)), \frac{1}{|2|^{n\beta}}t\right)\right) \right\} \\ & = \frac{t}{t + L^n\phi(x, y)} \rightarrow 1 \end{aligned}$$

as $n \rightarrow \infty$. Thus $Q(x) := \text{N-lim}_{n \rightarrow \infty} T^n(f)(x)$ is a quadratic mapping. The uniqueness of the quadratic mapping follows from (3) in Theorem 2.8. \square

COROLLARY 3.5. *Let $\theta \geq 0$, $p > 1$ be real numbers and let β be a real number with $\beta < \frac{p-1}{2}$. Let X be a normed linear space with norm $\|\cdot\|$. Suppose $f : X \rightarrow Y$ is a mapping satisfying*

$$(3.18) \quad N(D_\sigma f(x, y), t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}$$

for all $x, y \in X$ and $t > 0$. Then there exist a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{(2^p - 2^{2\beta+1})t}{(2^p - 2^{2\beta+1})t + 4\theta\|x\|^p}$$

where $\|x + \sigma(x)\|^p \leq 2^{p+\beta}\|x\|^p$ for all $x \in X$ and all $t > 0$.

Proof. Let $\phi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ in the Theorem 3.4 and let $L = 2^{2\beta-(p+1)}$. We have $0 < L < 1$. The remains follows from Theorem 3.4. \square

COROLLARY 3.6. *Let $\theta \geq 0$, $p \geq 1$ and β be real numbers with $\beta < p$. Let X be a normed linear space with norm $\|\cdot\|$. Suppose $f : X \rightarrow Y$ is a mapping satisfying*

$$(3.19) \quad N(D_\sigma f(x, y), t) \geq \frac{t}{t + \theta\|x\|^p\|y\|^p}$$

for all $x, y \in X$ and $t > 0$. Then there exist a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{(2^p - 2^{2\beta+1})t}{(2^p - 2^{2\beta+1})t + 4\theta \|x\|^p}$$

where $\|x + \sigma(x)\|^p \leq 2^{2p+\beta}\|x\|^p$ for all $x \in X$ and all $t > 0$.

Proof. Let $\phi(x, y) = \theta\|x\|^p\|y\|^p$ for all $x, y \in X$ in the Theorem 3.4. Let $L = 2^{2\beta-2p}$. Since $L = 2^{2\beta-2p}$ and $\|x + \sigma(x)\|^p \leq 2^{2p+\beta}\|x\|^p$, then we have $0 < L < 1$ and

$$\begin{aligned}\phi\left(\frac{x}{2}, \frac{x}{2}\right) &= \frac{1}{2^{2p}}\theta\|x\|^{2p} \leq \frac{L}{2^{2\beta}}\phi(x, x) \\ \phi(x + \sigma(x), x + \sigma(x)) &= \theta\|x + \sigma(x)\|^{2p} \leq 2^\beta\phi(2x, 2x)\end{aligned}$$

for all $x \in X$. The remains follows from Theorem 3.4. \square

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