

ON AP-HENSTOCK-STIELTJES INTEGRAL FOR FUZZY-NUMBER-VALUED FUNCTIONS

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ABSTRACT. In this paper we introduce the AP-Henstock-Stieltjes integral for fuzzy-number-valued functions which is an extension of the Henstock-Stieltjes integral and investigate some properties.

1. Introduction and Preliminaries

It is well-known that the Henstock integral for real valued function was first defined by Henstock [3, 4] in 1963. The Henstock integral is more powerful and simpler than the Lebesgue, Feynman integrals.

In 2012, Z. Gong and L. Wang introduced the concept of the Henstock-Stieltjes integrals of fuzzy-number-valued functions and obtained some properties [2].

In this paper we introduce the concept of the AP-Henstock-Stieltjes integral of fuzzy-number-valued functions and investigate some properties.

A Henstock partition of $[a, b]$ is a finite collection $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ such that $\{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ is a non-overlapping family of subintervals of $[a, b]$ covering $[a, b]$ and $\xi_i \in [x_{i-1}, x_i]$ for each $1 \leq i \leq n$. A gauge on $[a, b]$ is a function $\delta : [a, b] \rightarrow (0, \infty)$. A Henstock partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ is said to be δ -fine on $[a, b]$ if $[x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for each $1 \leq i \leq n$.

Let α be an increasing function on $[a, b]$. A function $f : [a, b] \rightarrow R$ is said to be Henstock-Stieltjes integrable to $L \in R$ with respect to α on $[a, b]$ if for every $\epsilon > 0$ there exists a positive function δ on $[a, b]$ such that $|\sum_{i=1}^n f(\xi_i)(\alpha(v_i) - \alpha(u_i)) - L| < \epsilon$ whenever $P = \{([u_i, v_i], \xi_i) : 1 \leq i \leq n\}$ is a δ -fine Henstock partition of $[a, b]$. We denote this fact as $(HS) \int_a^b f(x) d\alpha = L$ and $f \in HS[a, b]$. The function f is

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Henstock-Stieltjes integrable with respect to α on a set $E \subset [a, b]$ if f_{χ_E} is Henstock-Stieltjes integrable with respect to α on $[a, b]$, where χ_E denotes the characteristic function of E .

Fuzzy set $u : R \rightarrow [0, 1]$ is called a fuzzy number if u is a normal, convex fuzzy set, upper semi-continuous and $\text{supp } u = \overline{\{x \in R | u(x) > 0\}}$ is compact. Here $\overline{A}$ denotes the closure of A . We use E^1 to denote the fuzzy number space [4].

Let $u, v \in E^1, k \in R$, the addition and scalar multiplication are defined by

$$[u + v]_\lambda = [u]_\lambda + [v]_\lambda, [ku]_\lambda = k[u]_\lambda,$$

where $[u]_\lambda = \{x : u(x) \geq \lambda\} = [u_\lambda^-, u_\lambda^+]$ for any $\lambda \in [0, 1]$.

We use the Hausdorff distance between fuzzy numbers given by $D : E^1 \times E^1 \rightarrow [0, \infty)$ as follows

$$D(u, v) = \sup_{\lambda \in [0, 1]} d([u]_\lambda, [v]_\lambda) = \sup_{\lambda \in [0, 1]} \max\{|u_\lambda^- - v_\lambda^-|, |u_\lambda^+ - v_\lambda^+|\},$$

where d is the Hausdorff metric. $D(u, v)$ is called the distance between u and v .

LEMMA 1.1. [2] *If $u \in E^1$, then*

- (1) $[u]_\lambda$ is non-empty bounded closed interval for all $\lambda \in [0, 1]$.
- (2) $[u]_{\lambda_1} \supset [u]_{\lambda_2}$ for any $0 \leq \lambda_1 \leq \lambda_2 \leq 1$.
- (3) for any $\{\lambda_n\}$ converging increasingly to $\lambda \in (0, 1]$,

$$\bigcap_{n=1}^{\infty} [u]_{\lambda_n} = [u]_\lambda.$$

Conversely, if for all $\lambda \in [0, 1]$, there exists $A_\lambda \subset R$ satisfying (1) $\sim$ (3), then there exists a unique $u \in E^1$ such that $[u]_\lambda = A_\lambda, \lambda \in (0, 1]$, and $[u]_0 = \overline{\cup_{\lambda \in (0, 1]} [u]_\lambda} \subset A_0$.

DEFINITION 1.2. [2] Let α be an increasing function on $[a, b]$. A fuzzy number-valued function F is Henstock-Stieltjes integrable with respect to α on $[a, b]$ if there exists a fuzzy number $K \in E^1$ such that for every $\epsilon > 0$ there exists a positive function $\delta(x)$ such that

$$D\left(\sum_{i=1}^n F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), K\right) < \epsilon$$

whenever $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ is a δ -fine Henstock partition of $[a, b]$. We denote this fact as $(FHS) \int_a^b F(x) d\alpha = K$ and $(F, \alpha) \in FHS[a, b]$.

The fuzzy number-valued function F is Henstock-Stieltjes integrable with respect to α on a set $E \subset [a, b]$ if F_{χ_E} is Henstock-Stieltjes integrable with respect to α on $[a, b]$, where χ_E denotes the characteristic function of E .

2. The fuzzy number-valued AP-Henstock-Stieltjes Integral

In this section, we will define the fuzzy number-valued AP-Henstock-Stieltjes integral, which is an extension of the fuzzy number-valued Henstock-Stieltjes integral [4] and investigate some of their properties.

Let E be a measurable set and let x be a real number. The density of E at x is defined by

$$d_x E = \lim_{h \rightarrow 0+} \frac{\mu(E \cap (x - h, x + h))}{2h},$$

provided the limit exists. The point x is called a point of density of E if $d_x E = 1$. The E^d represents the set of all $x \in E$ such that x is a point of density of E .

An approximate neighborhood (or ad-nbd) of $x \in [a, b]$ is a measurable set $S_x \subset [a, b]$ containing x as a point of density. For every $x \in E \subset [a, b]$, choose an ad-nbd $S_x \subset [a, b]$ of x . then we say that $S = \{S_x : x \in E\}$ is a choice on E . A tagged interval $([u, v], x)$ is said to be fine to the choice $S = \{S_x\}$ if $u, v \in S_x$. Let $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ be a finite collection of non-overlapping tagged intervals. If $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ is fine to a choice S for each i , then we say that P is S -fine. Let $E \subset [a, b]$. If P is S -fine and each $\xi_i \in E$, then P is called S -fine on E . If P is S -fine and $[a, b] = \cup_{i=1}^n [u_i, v_i]$, then we say that P is S -fine partition of $[a, b]$.

DEFINITION 2.1. [6] A function $f : [a, b] \rightarrow R$ is AP-Henstock integrable if there exists a real number $A \in R$ such that for each $\epsilon > 0$ there is a choice S on $[a, b]$ such that

$$\left| \sum_{i=1}^n f(\xi_i)(v_i - u_i) - A \right| < \epsilon$$

for each S -fine partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ of $[a, b]$. In this case, A is called AP-Henstock integral of f on $[a, b]$, and we write $A = (APH) \int_a^b f$.

DEFINITION 2.2. Let α be an increasing function on $[a, b]$. A fuzzy number-valued function F is AP-Henstock-Stieltjes integrable with respect to α on $[a, b]$ if there exists a fuzzy number $K \in E^1$ such that for every $\epsilon > 0$ there exists a choice S on $[a, b]$ such that

$$D \left(\sum_{i=1}^n F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), K \right) < \epsilon$$

whenever $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ is a S -fine Henstock partition of $[a, b]$. We write $(APFHS) \int_a^b F(x) d\alpha = K$ and $(F, \alpha) \in APFHS[a, b]$.

The fuzzy number-valued function F is AP-Henstock-Stieltjes integrable with respect to α on a set $E \subset [a, b]$ if F_{χ_E} is AP-Henstock-Stieltjes integrable with respect to α on $[a, b]$, where χ_E denotes the characteristic function of E .

REMARK 2.3. If $F \in APFHS[a, b]$, then the integral value is unique.

From the definition of the fuzzy number-valued AP-Henstock-Stieltjes integral and the fact that (E^1, D) is a complete metric space, we can easily obtain the following theorem.

THEOREM 2.4. Let α be an increasing function on $[a, b]$. A fuzzy number-valued function F is AP-Henstock-Stieltjes integrable with respect to α on $[a, b]$ if and only if for every $\epsilon > 0$ there exists a choice S on $[a, b]$ such that for any S -fine partitions $P = \{([u, v], \xi)\}$ and $P' = \{([u', v'], \xi')\}$, we have

$$D \left(\sum_P F(\xi)(\alpha(v) - \alpha(u)), \sum_{P'} F(\xi')(\alpha(v') - \alpha(u')) \right) < \epsilon.$$

THEOREM 2.5. Let α be an increasing function on $[a, b]$ and let F be a fuzzy number-valued function on $[a, b]$. Then the following statements are equivalent:

- (1) $(F, \alpha) \in APFHS[a, b]$ and $(APFHS) \int_a^b F(x) d\alpha = K$.
- (2) for any $\lambda \in [0, 1]$, F_λ^- and F_λ^+ are AP-Henstock-Stieltjes integrable functions with respect to α on $[a, b]$ for and $\lambda \in [0, 1]$ uniformly (Choice S is independent of $\lambda \in [0, 1]$) and

$$\begin{aligned} & \left[(APFHS) \int_a^b F(x) d\alpha \right]_\lambda \\ &= \left[(APHS) \int_a^b F_\lambda^-(x) d\alpha, (APHS) \int_a^b F_\lambda^+(x) d\alpha \right]. \end{aligned}$$

Proof. (1) implies (2) : Since $(APFHS) \int_a^b F(x) d\alpha = K$, then for any $\epsilon > 0$, there exists a choice S on $[a, b]$ such that for any S -fine partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$, we have

$$D \left(\sum_{i=1}^n F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), K \right) < \epsilon.$$

By definition of D , for any S -fine partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ on $[a, b]$

$$\sup_{\lambda \in [0,1]} \max \left\{ \left| \left[\sum_{i=1}^n F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) \right]_{\lambda}^{-} - K_{\lambda}^{-} \right|, \right. \\ \left. \left| \left[\sum_{i=1}^n F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) \right]_{\lambda}^{+} - K_{\lambda}^{+} \right| \right\} < \epsilon.$$

Hence, we have

$$\left| \sum_{i=1}^n F_{\lambda}^{-}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) - K_{\lambda}^{-} \right| < \epsilon, \\ \left| \sum_{i=1}^n F_{\lambda}^{+}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) - K_{\lambda}^{+} \right| < \epsilon.$$

Therefore, $F_{\lambda}^{-}(x)$ and $F_{\lambda}^{+}(x)$ are AP-Henstock-Stieltjes with respect to α on $[a, b]$ for any $\lambda \in [0, 1]$, and

$$(APHS) \int_a^b F_{\lambda}^{-}(x) d\alpha = K_{\lambda}^{-}, \quad (APHS) \int_a^b F_{\lambda}^{+}(x) d\alpha = K_{\lambda}^{+}.$$

(2) implies (1) : Since $F_{\lambda}^{-}(x)$ and $F_{\lambda}^{+}(x)$ are AP-Henstock-Stieltjes integrable functions with respect to α on $[a, b]$ for any $\lambda \in [0, 1]$, For $\epsilon > 0$, there exists a choice S on $[a, b]$ such that for any S -fine partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ on $[a, b]$ and for any $\lambda \in [0, 1]$, we have

$$\left| \sum_{i=1}^n F_{\lambda}^{-}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) - K_{\lambda}^{-} \right| < \epsilon, \\ \left| \sum_{i=1}^n F_{\lambda}^{+}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) - K_{\lambda}^{+} \right| < \epsilon.$$

Then we can prove that $\{[K_{\lambda}^{-}, K_{\lambda}^{+}] : \lambda \in [0, 1]\}$ satisfies the conditions of Lemma 1.1. By the Lemma 1.1, $\{[K_{\lambda}^{-}, K_{\lambda}^{+}] : \lambda \in [0, 1]\}$ determines

a fuzzy number K . By the definition of D , for any S -fine partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ on $[a, b]$, we obtain

$$D\left(\sum_{i=1}^n F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), K\right) < 2\epsilon$$

Thus, $(F, \alpha) \in APFHS[a, b]$ and $(APFHS) \int_a^b F(x) d\alpha = K$. $\square$

From theorem 2.5 and the properties of AP-Henstock-Stieltjes integral, the followings theorems are obvious.

THEOREM 2.6. *Let $F, G \in APFHS[a, b]$ and $\beta, \gamma \in R$. Then*

(1) $\beta F + \gamma G \in APFHS[a, b]$ and

$$\begin{aligned} (APFHS) \int_a^b (\beta F + \gamma G) d\alpha \\ = \beta (APFHS) \int_a^b F d\alpha + \gamma (APFHS) \int_a^b G d\alpha. \end{aligned}$$

(2) If $F(x) \leq G(x)$ in $[a, b]$, then

$$(APFHS) \int_a^b F d\alpha \leq (APFHS) \int_a^b G d\alpha.$$

THEOREM 2.7. *Let $F \in APFHS[a, c]$ and $F \in APFHS[c, b]$. Then $F \in APFHS[a, b]$ and $(APFHS) \int_a^b F d\alpha = (APFHS) \int_a^c F d\alpha + (APFHS) \int_c^b F d\alpha$.*

THEOREM 2.8. *Let $F, G \in APFHS[a, b]$ and $D(F, G)$ is Lebesgue-Stieltjes integrable on $[a, b]$. Then*

$$D\left((APFHS) \int_a^b F d\alpha, (APFHS) \int_a^b G d\alpha\right) \leq (L) \int_a^b D(F, G) d\alpha.$$

Proof. By definition of distance,

$$\begin{aligned} D\left((APFHS) \int_a^b F d\alpha, (APFHS) \int_a^b G d\alpha\right) \\ = \sup_{\lambda \in [0, 1]} \max \left(\left| [((APHS) \int_a^b F d\alpha)]_{\lambda}^{-} - [((APHS) \int_a^b G d\alpha)]_{\lambda}^{-} \right|, \right. \\ \left. \left| [((APHS) \int_a^b F d\alpha)]_{\lambda}^{+} - [((APHS) \int_a^b G d\alpha)]_{\lambda}^{+} \right| \right) \\ = \sup_{\lambda \in [0, 1]} \max \left(\left| (APHS) \int_a^b ([F]_{\lambda}^{-} - [G]_{\lambda}^{-}) d\alpha \right|, \right. \end{aligned}$$

$$\begin{aligned}
 & \left| (APHS) \int_a^b ([F]_\lambda^+ - [G]_\lambda^+) d\alpha \right| \\
 & \leq \sup_{\lambda \in [0,1]} \max \left((L) \int_a^b |[F]_\lambda^- - [G]_\lambda^-| d\alpha, (L) \int_a^b |[F]_\lambda^+ - [G]_\lambda^+| d\alpha \right) \\
 & \leq (LS) \int_a^b D(F, G) d\alpha.
 \end{aligned}$$

This completes the proof. $\square$

THEOREM 2.9. *Let α be an increasing function on $[a, b]$ with $\alpha \in C^1[a, b]$ and let F be a bounded fuzzy number-valued on $[a, b]$. Then F is AP-Henstock-Stieltjes integrable with respect to α on $[a, b]$ if and only if $f\alpha'$ is AP-Henstock integrable on $[a, b]$. In this case, we have*

$$(APFHS) \int_a^b F(x) d\alpha = (APFH) \int_a^b F(x) \alpha'(x) dx,$$

where $(APFH)$ integral denotes the fuzzy number-valued Henstock AP-integral.

Proof. Since F is bounded, $\sup_{x \in [a,b]} D(F(x), 0)$ exists. Also since α' is uniformly continuous on $[a, b]$, for any $\epsilon > 0$ there exists $\eta > 0$ such that

$$|\alpha'(x) - \alpha'(y)| < \frac{\epsilon}{3 \sup_{x \in [a,b]} D(F(x), 0)(b-a)}$$

for any $x, y \in [a, b]$ satisfying $|x - y| < \eta$. Define a choice $S_1 = \{S_x^1\}$ on $[a, b]$ with Lebesgue measure $|S_x^1| < \eta$ for all $x \in [a, b]$.

Let $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ be a S_1 -fine partition on $[a, b]$, then by Lagrange mean value theorem, there exists $t_i \in (x_{i-1}, x_i)$ such that $\alpha(x_i) - \alpha(x_{i-1}) = \alpha'(t_i)(x_i - x_{i-1})$ ($1 \leq i \leq n$). Since $|t_i - x_i| \leq |S_{\xi_i}| < \eta$ for $1 \leq i \leq n$, we have

$$|\alpha'(t_i) - \alpha'(x_i)| < \frac{\epsilon}{3 \sup_{x \in [a,b]} D(F(x), 0)(b-a)}$$

for $1 \leq i \leq n$. Hence, we have

$$\begin{aligned}
 & D\left(\sum_{i=1}^n F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), \sum_{i=1}^n F(\xi_i)\alpha'(\xi_i)(x_i - x_{i-1})\right) \\
 & = D\left(\sum_{i=1}^n F(\xi_i)\alpha'(t_i)(x_i - x_{i-1}), \sum_{i=1}^n F(\xi_i)\alpha'(\xi_i)(x_i - x_{i-1})\right)
 \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=1}^n D\left(F(\xi_i)\alpha'(t_i)(x_i - x_{i-1}), F(\xi_i)\alpha'(\xi_i)(x_i - x_{i-1})\right) \\
&= \sum_{i=1}^n \sup_{\lambda \in [0,1]} \max \left\{ |F_{\lambda}^{-}(\xi_i)(\alpha'(t_i) - \alpha'(\xi_i))|, \right. \\
&\quad \left. |F_{\lambda}^{+}(\xi_i)(\alpha'(t_i) - \alpha'(\xi_i))| \right\} (x_i - x_{i-1}) \\
&\leq \sum_{i=1}^n |\alpha'(t_i) - \alpha'(\xi_i)| (x_i - x_{i-1}) \sup_{\lambda \in [0,1]} \max\{|F_{\lambda}^{-}(\xi_i)|, |f_{\lambda}^{+}(\xi_i)|\} \\
&\leq (b-a) \frac{\epsilon}{3 \sup_{x \in [a,b]} D(F(x), 0)(b-a)} \sup_{x \in [a,b]} D(F(x), 0) \leq \frac{\epsilon}{3}.
\end{aligned}$$

On the other hand, since F is fuzzy number-valued Henstock-stieltjes integrable with respect to α on $[a, b]$, by Theorem 2.4 there is a choice S_2 on $[a, b]$ such that for any S_2 -fine partitions $P = \{[u, v], \xi\}$ and $P' = \{[u', v'], \xi'\}$, we have

$$D\left(\sum_P f(\xi)(\alpha(v) - \alpha(u)), \sum_{P'} f(\xi')(\alpha(v') - \alpha(u'))\right) < \frac{\epsilon}{3}.$$

Define a choice $S_3 = \{S_x^1 \cap S_x^2 : S_x^1 \in S_1, S_x^2 \in S_2\}$. Let $P = \{[u, v], \xi\}$ and $P' = \{[u', v'], \xi'\}$ be S_3 -fine partitions on $[a, b]$. we have

$$\begin{aligned}
&D\left(\sum_P F(\xi)\alpha'(\xi)(v - u), \sum_{P'} F(\xi')\alpha'(\xi')(v' - u')\right) \\
&\leq D\left(\sum_P F(\xi)\alpha'(\xi)(v - u), \sum_P F(\xi)(\alpha(v) - \alpha(u))\right) \\
&\quad + D\left(\sum_P F(\xi)(\alpha(v) - \alpha(u)), \sum_{P'} F(\xi')\alpha(v') - \alpha(u')\right) \\
&\quad + D\left(\sum_{P'} F(\xi')\alpha(v') - \alpha(u'), \sum_{P'} F(\xi')\alpha'(\xi')(v' - u')\right) < \epsilon.
\end{aligned}$$

Hence, $F\alpha'$ is AP-Henstock integrable on $[a, b]$.

Conversely, if $F\alpha'$ is AP-Henstock integrable on $[a, b]$, then for any $\epsilon > 0$, there is a choice $S_3 = \{S_x^3\}$ on $[a, b]$ such that for any S_3 -fine partitions $P = \{[u, v], \xi\}$ and $P' = \{[u', v'], \xi'\}$, we have

$$D\left(\sum_P F(\xi)\alpha'(\xi)(v - u), \sum_{P'} F(\xi')\alpha'(\xi')(v' - u')\right) < \frac{\epsilon}{3}.$$

For a choice $S_4 = \{S_x^1 \cap S_x^3 : S_x^1 \in S_1, S_x^3 \in S_3\}$, let $P = \{[u, v], \xi\}$ and $P' = \{[u', v'], \xi'\}$ be S_4 -fine partitions on $[a, b]$. then we have

$$\begin{aligned} & D\left(\sum_P F(\xi)(\alpha(v) - \alpha(u)), \sum_{P'} F(\xi')(\alpha(v') - \alpha(u'))\right) \\ & \leq D\left(\sum_P F(\xi)(\alpha(v) - \alpha(u)), \sum_P F(\xi)\alpha'(\xi)(v - u)\right) \\ & \quad + D\left(\sum_P F(\xi)\alpha'(\xi)(v - u), \sum_{P'} F(\xi')\alpha'(\xi')(v' - u')\right) \\ & \quad + D\left(\sum_{P'} F(\xi')\alpha'(\xi')(v' - u'), \sum_{P'} F(\xi')(\alpha(v') - \alpha(u'))\right) < \epsilon. \end{aligned}$$

Hence, F is fuzzy number-valued Henstock-Stieltjes integrable with respect to α on $[a, b]$.

Next, we will show that

$$(APFHS) \int_a^b F(x) d\alpha = (APFH) \int_a^b F(x) \alpha'(x) dx.$$

For any S -fine partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ on $[a, b]$, by the Lagrange mean value theorem there exists $t_i \in (x_i - x_{i-1})$ such that

$$\alpha(x_i) - \alpha(x_{i-1}) = \alpha'(t_i)(x_i - x_{i-1}) (1 \leq i \leq n).$$

Hence, we have

$$\sum_{i=1}^n F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) = \sum_{i=1}^n F(\xi_i)\alpha'(t_i)(x_i - x_{i-1}).$$

Since F and $F\alpha'$ are bounded functions on $[a, b]$, we have

$$(APFHS) \int_a^b F(x) d\alpha = (APFH) \int_a^b F(x) \alpha'(x) dx.$$

This completes the proof. $\square$

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