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DERIVATION OF THE g-NAVIER-STOKES EQUATIONS

Jaiok Roh*

Abstract. The 2D g-Navier-Stokes equations are a certain modified Navier-Stokes equations and have the following form,

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}, \quad \text{in } \Omega$$

with the continuity equation

$$\nabla \cdot (g\mathbf{u}) = 0, \quad \text{in } \Omega,$$

where g is a suitable smooth real valued function. In this paper, we will derive 2D g-Navier-Stokes equations from 3D Navier-Stokes equations. In addition, we will see the relationship between two equations.

1. Introduction

By concerning the reaction-diffusion and damped wave equations on thin domains, Hale and Raugel([1], [2], [3]) originated the study of the Navier-Stokes equations on thin domains.

In [4] and [5], Raugel and Sell proved global existence of strong solutions for large initial data and forcing terms in thin three dimensional domains for the purely periodic boundary conditions and the periodic-Dirichlet boundary conditions, that is, periodic conditions in the thin vertical direction and homogeneous Dirichlet conditions on the lateral boundary condition $\Gamma_l = \partial\Omega \times (0, \epsilon)$, where $\Omega \subset R^2$.

An essential tool in their proof is the vertical mean operator M, which allows the decomposition of every function $\mathbf{U}$ on $\Omega_\epsilon = \Omega \times (0, \epsilon)$ into the sum of a function $M\mathbf{U} = \mathbf{v}(x_1, x_2)$ which does not depend on the vertical variable, and a function $(I - M)\mathbf{U} = \mathbf{w}(x_1, x_2, x_3)$, with vanishing vertical mean and thus to use more precise Sobolev and Poincaré

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inequalities. Then, they showed that the reduced 3D Navier-Stokes evolutionary equations by $\mathbf{v}$ incorporates the 2D Navier-Stokes equations on Ω. Later, by using same tool as Raugel and Sell with improved Agmon inequalities, Temam and Ziane([6], [7]) generalized the results of ([4], [5]) to other boundary conditions and, in the case of the free boundary conditions, to thin spherical domains.

In this paper, we apply Raugel and Sell methods on $\Omega_g = \Omega_2 \times (0, g)$, where Ω_2 is a bounded region in the plane and $g = g(x_1, x_2)$ is a smooth function defined on Ω_2 with $0 < m \le g(x_1, x_2) \le M$, for $(x_1, x_2) \in \Omega_2$. And we derive the 2D g-Navier-Stokes equations from 3D Navier-Stokes equations.

2. Main Theorems

Now, we consider $3D$ Navier-Stokes equations,

$$\begin{aligned} \frac{\partial \mathbf{U}}{\partial t} - \nu\Delta\mathbf{U} + (\mathbf{U}\cdot\nabla)\mathbf{U} + \nabla\Phi &= \mathbf{F}, \ \text{ in } \ \Omega_g \\ \nabla\cdot\mathbf{U} &= 0, \ \text{ in } \ \Omega_g, \end{aligned}$$

with the boundary condition

$$\mathbf{U}\cdot\mathbf{n} = 0 \quad \text{on} \qquad \partial_{top}\Omega_g \cup \partial_{bottom}\Omega_g \tag{1}$$

where

$$\begin{aligned} \partial_{top}\Omega_g &= \{(y_1, y_2, y_3) \in \Omega_g \ : \ y_3 = g(y_1, y_2)\}, \\ \partial_{bottom}\Omega_g &= \{(y_1, y_2, y_3) \in \Omega_g \ : \ y_3 = 0\}. \end{aligned}$$

The lateral boundary condition corresponding to $\partial\Omega_2$ does not affect to the derivation of the 2D g-Navier-Stokes equations. But, in this paper we consider the periodic and Dirichlet boundary conditions to study the 2D g-Navier-Stokes equations.

Now we define $\mathbf{v}(y_1, y_2) = (\mathbf{v}_1(y_1, y_2), \mathbf{v}_2(y_1, y_2), \mathbf{v}_3(y_1, y_2))$ as

$$\mathbf{v}_i(y_1, y_2) = M\mathbf{U}_i(y_1, y_2, y_3) = \frac{1}{g(y_1, y_2)} \int_0^{g(y_1, y_2)} \mathbf{U}_i(y_1, y_2, y_3)\ dy_3,$$

where $\mathbf{U} = (\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3)$, for $i = 1, 2, 3$. Now, for $\mathbf{w} = (\mathbf{v}_1, \mathbf{v}_2)$, we get the following theorem.

THEOREM 2.1. *Assume that $\nabla\cdot\mathbf{U} = 0$ in Ω_g and that (1) is valid. Then one has*

$$\nabla_2\cdot(g\mathbf{w}) = \frac{\partial(g\mathbf{v}_1)}{\partial x_1} + \frac{\partial(g\mathbf{v}_2)}{\partial x_2} = \nabla g\cdot\mathbf{w} + g\ (\nabla_2\cdot\mathbf{w}) = 0 \ \text{ in } \Omega_2,$$

where $\nabla_2 = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$ *and* $\nabla g = (\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2})$.

Proof. First we consider the change of variables

$$y_1 = x_1,\ y_2 = x_2,\ y_3 = x_3 g(x_1, x_2)$$

maps Ω_3 onto Ω_g, where $\Omega_3 = \Omega_2 \times (0, 1)$. Then we obtain from the chain rule that

$$\frac{\partial x_3}{\partial y_1} = -\frac{y_3}{g^2(y_1, y_2)} \times \frac{\partial g}{\partial y_1} = -\frac{x_3}{g} \times \frac{\partial g}{\partial x_1} \text{ and } \frac{\partial x_3}{\partial y_2} = -\frac{x_3}{g} \times \frac{\partial g}{\partial x_2}.$$

Also, we have for $\mathbf{u}(x_1, x_2, x_3) = \mathbf{U}(y_1, y_2, y_3)$,

$$\frac{\partial \mathbf{U}}{\partial y_1} = \frac{\partial \mathbf{u}}{\partial x_1} + \frac{\partial \mathbf{u}}{\partial x_3} \times \frac{\partial x_3}{\partial y_1} = \frac{\partial \mathbf{u}}{\partial x_1} - \frac{\partial \mathbf{u}}{\partial x_3}(\frac{x_3}{g} \times \frac{\partial g}{\partial x_1})$$
$$\frac{\partial \mathbf{U}}{\partial y_2} = \frac{\partial \mathbf{u}}{\partial x_2} - \frac{\partial \mathbf{u}}{\partial x_3}(\frac{x_3}{g} \times \frac{\partial g}{\partial x_2}), \quad \frac{\partial \mathbf{U}}{\partial y_3} = \frac{\partial \mathbf{u}}{\partial x_3}(\frac{\partial x_3}{\partial y_3}) = (\frac{1}{g}\frac{\partial \mathbf{u}}{\partial x_3}).$$

Therefore we have

$$(2) \qquad \nabla \cdot \mathbf{U} = [\frac{\partial \mathbf{u}_1}{\partial x_1} + \frac{\partial \mathbf{u}_2}{\partial x_2} + \frac{1}{g}\frac{\partial \mathbf{u}_3}{\partial x_3} - \frac{x_3}{g}(\frac{\partial \mathbf{u}_1}{\partial x_3}\frac{\partial g}{\partial x_1} + \frac{\partial \mathbf{u}_2}{\partial x_3}\frac{\partial g}{\partial x_2})].$$

Now we note

$$\mathbf{v}_i(x_1, x_2) = \frac{1}{g(y_1, y_2)} \int_0^{g(y_1,y_2)} \mathbf{U}_i(y_1, y_2, y_3)\, dy_3 = \int_0^1 \mathbf{u}_i(x_1, x_2, x_3)\, dx_3,$$

to obtain the followings:

$$\int_0^1 g\frac{\partial \mathbf{u}_1}{\partial x_1}\ dx_3 = g\frac{\partial \mathbf{v}_1}{\partial x_1}, \quad \int_0^1 g\frac{\partial \mathbf{u}_2}{\partial x_2}\ dx_3 = g\frac{\partial \mathbf{v}_2}{\partial x_2}$$
$$\int_0^1 \frac{\partial \mathbf{u}_3}{\partial x_3}\ dx_3 = \mathbf{u}_3(x_1, x_2, 1) - \mathbf{u}_3(x_1, x_2, 0)$$
$$-\int_0^1 x_3\frac{\partial \mathbf{u}_1}{\partial x_3}\frac{\partial g}{\partial x_1}\ dx_3 = -\frac{\partial g}{\partial x_1}\int_0^1 x_3\frac{\partial \mathbf{u}_1}{\partial x_3}\ dx_3$$
$$= \frac{\partial g}{\partial x_1}[\int_0^1 \mathbf{u}_1\ dx_3] - \frac{\partial g}{\partial x_1}x_3\mathbf{u}_1|_0^1$$
$$= \mathbf{v}_1\frac{\partial g}{\partial x_1} - \frac{\partial g}{\partial x_1}\mathbf{u}_1(x_1, x_2, 1)$$
$$-\int_0^1 x_3\frac{\partial \mathbf{u}_2}{\partial x_3}\frac{\partial g}{\partial x_2}\ dx_3 = \mathbf{v}_2\frac{\partial g}{\partial x_2} - \frac{\partial g}{\partial x_2}\mathbf{u}_2(x_1, x_2, 1).$$

Thus, we have

$$0 = \int_0^{g(y_1,y_2)} \nabla \cdot \mathbf{U}\ dy_3 = \int_0^1 (\nabla \cdot \mathbf{U})\ g\ dx_3$$

$$= g(\frac{\partial \mathbf{v}_1}{\partial x_1} + \frac{\partial \mathbf{v}_2}{\partial x_2}) + \mathbf{v}_1 \frac{\partial g}{\partial x_1} + \mathbf{v}_2 \frac{\partial g}{\partial x_2} + \mathrm{BC}, \tag{3}$$

where BC is the boundary conditions on Ω_g, i.e.,

$$BC = \mathbf{u}_3(x_1, x_2, 1) - \mathbf{u}_3(x_1, x_2, 0) - \frac{\partial g}{\partial x_1}\mathbf{u}_1(x_1, x_2, 1) - \frac{\partial g}{\partial x_2}\mathbf{u}_2(x_1, x_2, 1).$$

For the bottom part of Ω_g, the normal vector $\mathbf{n}$ is $\mathbf{n} = (0, 0, -1)$. Thus

$$\mathbf{U} \cdot \mathbf{n} = -\mathbf{U}_3|_{y_3=x_3=0} = -\mathbf{U}_3(y_1, y_2, 0) = -\mathbf{u}_3(x_1, x_2, 0) = 0.$$

For the top of Ω_g, one has $\mathbf{n} = \alpha(-\frac{\partial g}{\partial y_1}, -\frac{\partial g}{\partial y_2}, 1)$ where α is chosen so that $\| \mathbf{n} \| = 1$. So we have

$$\alpha^{-1}\ \mathbf{U} \cdot \mathbf{n}|_{top} = (-\frac{\partial g}{\partial y_1}\mathbf{U}_1 - \frac{\partial g}{\partial y_2}\mathbf{U}_2 + \mathbf{U}_3)|_{top}$$

$$= -\frac{\partial g}{\partial x_1}\mathbf{u}_1(x_1, x_2, 1) - \frac{\partial g}{\partial x_2}\mathbf{u}_2(x_1, x_2, 1) + \mathbf{u}_3(x_1, x_2, 1) = 0.$$

It then follows from assumption that BC = 0. This complete the proof by (3). □

Now, we assume that

$$\mathbf{U}(y_1, y_2, y_3) = (\mathbf{U}_1(y_1, y_2), \mathbf{U}_2(y_1, y_2), \mathbf{U}_3(y_1, y_2, y_3))$$
$$= (\mathbf{u}_1(x_1, x_2), \mathbf{u}_2(x_1, x_2), \mathbf{u}_3(x_1, x_2, x_3)) = \mathbf{u}(x_1, x_2, x_3).$$

Then, we raise the following questions:

1. What can we say about $\mathbf{u}_3(x_1, x_2, x_3) = \mathbf{U}_3(y_1, y_2, y_3)$ if $\nabla \cdot \mathbf{U} = 0$ in Ω_g?
2. What can we say about $\mathbf{u}_3(x_1, x_2, x_3) = \mathbf{U}_3(y_1, y_2, y_3)$ if $\mathbf{U} \cdot \mathbf{n} = 0$ on the top and bottom of Ω_g?

For the answer, we have the following theorem.

THEOREM 2.2. *Let* $\mathbf{U}(y_1, y_2, y_3) = (\mathbf{U}_1(y_1, y_2), \mathbf{U}_2(y_1, y_2), \mathbf{U}_3(y_1, y_2, y_3))$. *Then we have* $\nabla \cdot \mathbf{U} = 0$ *on* Ω_g *and*

$$\mathbf{U} \cdot \mathbf{n} = 0 \qquad \text{on the top and bottom of} \quad \Omega_g,$$

if and only if we obtain

$$\mathbf{u}_3(x_1, x_2, x_3) = x_3(\frac{\partial g}{\partial x_1}\mathbf{u}_1 + \frac{\partial g}{\partial x_2}\mathbf{u}_2) = -g\ x_3\ (\frac{\partial \mathbf{u}_1}{\partial x_1} + \frac{\partial \mathbf{u}_2}{\partial x_2}).$$

Proof. First we know that if $\nabla \cdot \mathbf{U} = 0$ then (2) implies

$$\frac{\partial \mathbf{u}_1}{\partial x_1} + \frac{\partial \mathbf{u}_2}{\partial x_2} + \frac{1}{g}\frac{\partial \mathbf{u}_3}{\partial x_3} = 0.$$

Thus we have

$$\frac{\partial \mathbf{u}_3}{\partial x_3} = -g(\frac{\partial \mathbf{u}_1}{\partial x_1} + \frac{\partial \mathbf{u}_2}{\partial x_2}),$$

which implies that

$$\mathbf{u}_3 = -x_3 \ g \ (\frac{\partial \mathbf{u}_1}{\partial x_1} + \frac{\partial \mathbf{u}_2}{\partial x_2}) + c(x_1, x_2),$$

for some function $c(x_1, x_2)$. Since $\mathbf{U} \cdot \mathbf{n} = 0$ on the bottom, one has $\mathbf{U}_3(y_1, y_2, 0) = \mathbf{u}_3(x_1, x_2, 0) = 0$, which implies that

$$c(x_1, x_2) = 0, \qquad \text{and} \qquad \mathbf{u}_3(x_1, x_2, x_3) = -x_3 \ g \ (\frac{\partial \mathbf{u}_1}{\partial x_1} + \frac{\partial \mathbf{u}_2}{\partial x_2}).$$

By the definition of $\mathbf{v}_i$, note $\mathbf{v}_i = \mathbf{u}_i$, for $i = 1, 2$. So, by theorem 2.1 we have $\nabla_2 \cdot g\mathbf{u} = \frac{\partial(g\mathbf{u}_1)}{\partial x_1} + \frac{\partial(g\mathbf{u}_2)}{\partial x_2} = 0$ and

$$\mathbf{u}_3(x_1, x_2, x_3) = x_3(\frac{\partial g}{\partial x_1}\mathbf{u}_1 + \frac{\partial g}{\partial x_2}\mathbf{u}_2).$$

The converse comes from a direct calculation. □

Now, let us go back to our problem, 3D Navier-Stokes equations on Ω_g,

$$\begin{aligned} \frac{\partial \mathbf{U}}{\partial t} - \nu\Delta\mathbf{U} + (\mathbf{U}\cdot\nabla)\mathbf{U} + \nabla\Phi &= \mathbf{F}, \ \text{in} \ \Omega_g \\ \nabla\cdot\mathbf{U} &= 0, \ \text{in} \ \Omega_g, \end{aligned}$$

with the boundary condition

$$\mathbf{U}\cdot\mathbf{n} = 0 \quad \text{on} \qquad \partial_{top}\Omega_g \cup \partial_{bottom}\Omega_g.$$

Since $(\mathbf{U}(y_1, y_2, y_3)) = (\mathbf{U}_1(y_1, y_2), \mathbf{U}_2(y_1, y_2), \mathbf{U}_3(y_1, y_2, y_3))$ we have

$$\mathbf{v}_i(x_1, x_2) = \mathbf{U}_i(y_1, y_2) = \mathbf{u}_i(x_1, x_2), \quad i = 1, 2.$$

Therefore, by theorem 2.1 and theorem 2.2, $\mathbf{w} = (\mathbf{u}_1, \mathbf{u}_2) = (\mathbf{U}_1, \mathbf{U}_2)$ satisfies the 2D g-Navier-Stokes equations,

$$\begin{aligned} \frac{\partial \mathbf{w}}{\partial t} - \nu\Delta\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{w} + \nabla p &= \mathbf{f}, \ \text{in} \ \Omega_2 \\ \nabla\cdot g\mathbf{w} &= 0, \ \text{in} \ \Omega_2, \end{aligned}$$

and third variable $\mathbf{U}_3(y_1, y_2, y_3) = \mathbf{u}_3(x_1, x_2, x_3)$ can be solved by $(\mathbf{U}_1, \mathbf{U}_2) = (\mathbf{u}_1, \mathbf{u}_2)$.

Therefore, we motivate to study 2D g-Navier-Stokes equations for 3D Navier-Stokes equations on thin domain Ω_g.

Remark 2.1. In theorem 2.1 and theorem 2.2, we do not use any boundary condition other than (2). If $\mathbf{U}$ is periodic in (y_1, y_2), i.e., $\mathbf{U}(0, y_2, y_3) = \mathbf{U}(1, y_2, y_3)$ and $\mathbf{U}(y_1, 0, y_3) = \mathbf{U}(y_1, 1, y_3)$, then $\mathbf{w}$ is also periodic in (y_1, y_2). Likewise, if $\mathbf{U}$ satisfies Dirichlet conditions for $(y_1, y_2) \in \partial\Omega_2$, then $\mathbf{w}$ does as well.

Also, since $\mathbf{u}_3(x_1, x_2, x_3) = x_3(\frac{\partial g}{\partial x_1}\mathbf{u}_1 + \frac{\partial g}{\partial x_2}\mathbf{u}_2)$, for smooth and bounded function $g(x_1, x_2)$, we have

$$\| \mathbf{U}_3 \|_{L^2(\Omega_g)} \leq \alpha \| \mathbf{w} \|_{L^2(\Omega_2)}, \quad \| \nabla\mathbf{U}_3 \|_{L^2(\Omega_g)} \leq \beta \| \mathbf{w} \|_{H^1(\Omega_2)},$$

for some positive constants α, β.

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Department of Mathematics
Hallym University
Chuncheon 200-702, Republic of Korea
E-mail: joroh@hallym.ac.kr