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ON CONVERGENCE THEOREMS OF THE AP-HENSTOCK-STIELTJES INTEGRAL FOR FUZZY NUMBER-VALUED FUNCTIONS

Ju Han Yoon*

Abstract. In this paper we introduce the concept of equi-integrability of sequence of the fuzzy number-valued AP-Henstock-Stieltjes integrable functions. Under this concept, we prove two convergence theorems for sequences of the fuzzy number-valued AP-Henstock-Stieltjes integrable functions.

1. Introduction and Preliminaries

That the Henstock integral of real valued function was first defined by Henstock in 1963 [2, 4]. It is well known that the Henstock integral is more powerful and simpler than the Lebesgue and Feynman integrals.

In 2018, J. H. Yoon introduced the AP-Henstock-Stieltjes integral of fuzzy number-valued functions and investigated some properties [9].

In this paper we introduce the concept of equi-integrability of sequence of the fuzzy number-valued AP-Henstock-Stieltjes integrable functions. Under this concept, we prove convergence theorems for sequences of the fuzzy number-valued AP-Henstock-Stieltjes integrable functions.

A Henstock partition of $[a, b]$ is a finite collection $P = \{([x_{i-1}, x_i], \xi_i) : 1 \le i \le n\}$ such that $\{([x_{i-1}, x_i], \xi_i) : 1 \le i \le n\}$ is a non-overlapping family of subintervals of $[a, b]$ covering $[a, b]$ and $\xi_i \in [x_{i-1}, x_i]$ for each $1 \le i \le n$. A gauge on $[a, b]$ is a function $\delta : [a, b] \to (0, \infty)$. A Henstock partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \le i \le n\}$ is said to be δ-fine on $[a, b]$ if $[x_{i-i}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for each $1 \le i \le n$.

Let α be an increasing function on $[a, b]$. A function $f : [a, b] \to R$ is said to be Henstock-Stieltjes integrable to $L \in R$ with respect to α on $[a, b]$ if for every $\epsilon > 0$ there exists a positive function δ on $[a, b]$ such that

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$|\sum_{i=1}^{n} f(\xi_i)(\alpha(v_i)-\alpha(u_i))-L| < \epsilon$ whenever $P=\{([u_i,v_i],\xi_i) : 1 \le i \le n\}$ is a δ -fine Henstock partition of $[a,b]$. We write $(HS)\int_a^b f(x)d\alpha = L$ and $f \in HS[a,b]$. The function f is Henstock-Stieltjes integrable with respect to α on a set $E \subset [a,b]$ if f_{χ_E} is Henstock-Stieltjes integrable with respect to α on $[a,b]$, where χ_E denotes the characteristic function of E.

Fuzzy set $u : R \to [0,1]$ is called a fuzzy number if u is a normal, convex fuzzy set, upper semi-continuous and supp $u = \overline{\{x \in R | u(x) > 0\}}$ is compact. Here $\overline{A}$ denotes the closure of A. We use E^1 to denote the fuzzy number space [5].

Let $u, v \in E^1, k \in R$. The addition and scalar multiplication are defined by

$$[u+v]_\lambda = [u]_\lambda + [v]_\lambda, \ [ku]_\lambda = k[u]_\lambda,$$

where $[u]_\lambda = \{x : u(x) \ge \lambda\} = [u^-_\lambda,\, u^+_\lambda]$ for any $\lambda \in [0,1]$.

We use the Hausdorff distance between fuzzy numbers given by $D : E^1 \times E^1 \to [0, \infty)$ as follows

$$D(u,v) = \sup_{\lambda\in[0,1]} d([u]_\lambda,\, [v]_\lambda) = \sup_{\lambda\in[0,1]} \max\{|u^-_\lambda - v^-_\lambda|, |u^+_\lambda - v^+_\lambda|\},$$

where d is the Hausdorff metric. $D(u,v)$ is called the distance between u and v.

LEMMA 1.1. [3] *If $u \in E^1$, then*

(1) $[u]_\lambda$ *is non-empty bounded closed interval for all* $\lambda \in [0,1]$.
(2) $[u]_{\lambda_1} \supset [u]_{\lambda_2}$ *for any* $0 \le \lambda_1 \le \lambda_2 \le 1$.
(3) *for any* $\{\lambda_n\}$ *converging increasingly to* $\lambda \in (0,1]$,

$$\bigcap_{n=1}^{\infty} [u]_{\lambda_n} = [u]_\lambda.$$

Conversely, if for all $\lambda \in [0,1]$, there exists $A_\lambda \subset R$ satisfying (1) $\sim$ (3), *then there exists a unique $u \in E^1$ such that $[u]_\lambda = A_\lambda$, $\lambda \in (0,1]$, and $[u]_0 = \overline{\cup_{\lambda\in(0,1]}[u]_\lambda} \subset A_0$.*

DEFINITION 1.2. [3] Let α be an increasing function on $[a,b]$. A fuzzy number-valued function F is Henstock-Stieltjes integrable with respect to α on $[a,b]$ if there exists a fuzzy number $K \in E^1$ such that for every $\epsilon > 0$ there exists a positive function $\delta(x)$ such that

$$D\left(\sum_{i=1}^{n} F(\xi_i)(\alpha(x_i)-\alpha(x_{i-1})),\ K\right) < \epsilon$$

whenever $P = \{([x_{i-1}, x_i], \xi_i) : 1 \le i \le n\}$ is a δ -fine Henstock partition of $[a, b]$. We write $(FHS) \int_a^b F(x)d\alpha = K$ and $(F, \alpha) \in FHS[a, b]$.

The fuzzy number-valued function F is Henstock-Stieltjes integrable with respect to α on a set $E \subset [a, b]$ if F_{χ_E} is Henstock-Stieltjes integrable with respect to α on $[a, b]$, where χ_E denotes the characteristic function of E.

2. Convergence theorems for sequence of the fuzzy number-valued AP-Henstock-Stieltjes integrable functions.

In this section, we define the concept of equi-integrablility for sequence of fuzzy number-valued AP-Henstock-Stieltjes integrable functions and prove two convergence theorems for sequences of the fuzzy number-valued AP-Henstock-Stieltjes integrable functions.

Let E be a measurable set and let x be a real number. The density of E at x is defined by

$$d_x E = \lim_{h \to 0+} \frac{\mu(E \cap (x - h, x + h))}{2h},$$

provided the limit exists. The point x is called a point of density of E if $d_x E = 1$. The E^d represents the set of all $x \in E$ such that x is a point of density of E.

An approximate neighborhood(or ap-nbd) of $x \in [a, b]$ is a measurable set $S_x \subset [a, b]$ containing x as a point of density. For every $x \in E \subset [a, b]$, choose an ap-nbd $S_x \subset [a, b]$ of x. Then we say that $S = \{S_x : x \in E\}$ is a choice on E. A tagged interval $([u, v], x)$ is said to fine to the choice $S = \{S_x\}$ if $u, v \in S_x$. Let $P = \{([x_{i-1}, x_i], \xi_i) : 1 \le i \le n\}$ be a finite collection of non-overlapping tagged intervals. If $P = \{([x_{i-1}, x_i], \xi_i) : 1 \le i \le n\}$ is fine to a choice S for each i, then we say that P is an S-fine. Let $E \subset [a, b]$. If P is S-fine and each $\xi_i \in E$, then P is called S-fine on E. If P is S-fine and $[a, b] = \cup_{i=1}^n [u_i, v_i]$, then we say that P is an S-fine Henstock partition of $[a, b]$. we denote that N is the set of natural numbers.

Definition 2.1. [8] A function $f : [a, b] \to R$ is AP-Henstock integrable if there exists a real number $A \in R$ such that for each $\epsilon > 0$ there is a choice S on $[a, b]$ such that

$$\left| \sum_{i=1}^n f(\xi_i)(v_i - u_i) - A \right| < \epsilon$$

for each S-fine Henstock partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \le i \le n\}$ of $[a, b]$. In this case, A is called the AP-Henstock integral of f on $[a, b]$, and we write $A = (APH) \int_a^b f$.

DEFINITION 2.2. [9] Let α be an increasing function on $[a, b]$. A fuzzy number valued function F is AP-Henstock-Stieltjes integrable with respect to α on $[a, b]$ if there exists a fuzzy number $K \in E^1$ such that for every $\epsilon > 0$ there exists a choice S on $[a, b]$ such that

$$D\left(\sum_{i=1}^{n} F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})),\, K\right) < \epsilon$$

whenever $P = \{([x_{i-1}, x_i], \xi_i) : 1 \le i \le n\}$ is an S-fine Henstock partition of $[a, b]$. We write $(APFHS) \int_a^b F(x) d\alpha = K$ and $(F, \alpha) \in APFHS[a, b]$.

The fuzzy number-valued function F is AP-Henstock-Stieltjes integrable with respect to α on a set $E \subset [a, b]$ if F_{χ_E} is AP-Henstock-Stieltjes integrable with respect to α on $[a, b]$, where χ_E denotes the characteristic function of E.

DEFINITION 2.3. Let $\alpha : [a, b] \to R$ be an increasing function and for each $n \in N$ let $F_n : [a, b] \to E^1$ be an AP-Henstock-Stieltjes integrable functions with respect to α. The sequence $\{F_n\}$ is equi-integrable with respect to α on $[a, b]$ if for every $\epsilon > 0$ there exists a choice S on $[a, b]$ such that for any S-fine Henstock partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \le i \le k\}$, we have

$$D\left(\sum_{i=1}^{k} F_n(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})),\, (APFHS) \int_a^b F_n(x) d\alpha\right) < \epsilon$$

for all $n \in N$.

DEFINITION 2.4. Let $\{\alpha_n\}$ be a sequence of increasing functions defined on $[a, b]$ and let $F : [a, b] \to E^1$ be an AP-Henstock-Stieltjes integrable function with respect to α_n on $[a, b]$ for every $n \in N$. The fuzzy number-valued function F is equi-integrable on $[a, b]$ with respect to the sequence $\{\alpha_n\}$ on $[a, b]$ if for every $\epsilon > 0$ there exists a choice S on $[a, b]$ such that for any S-fine Henstock partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \le i \le k\}$, we have

$$D\left(\sum_{i=1}^{k} F(\xi_i)(\alpha_n(x_i) - \alpha_n(x_{i-1})), (APFHS) \int_a^b F(x) d\alpha_n\right) < \epsilon$$

for all $n \in N$.

THEOREM 2.5. *Let $\{F_n\}$ be a sequence of fuzzy number-valued AP-Henstock-Stieltjes integrable functions with respect to α on $[a,b]$ and let $\{F_n\}$ converge pointwise to F on $[a,b]$. If $\{F_n\}$ is equi-integrable with respect to α on $[a,b]$, then F is AP-Henstock-Stieltjes integrable with respect to α on $[a,b]$ and*

$$(APFHS)\int_a^b F(x)\,d\alpha = \lim_{n\to\infty}(APFHS)\int_a^b F_n(x)\,d\alpha.$$

Proof. Since $\{F_n\}$ is a sequence of fuzzy number-valued AP-Henstock-Stieltjes integrable functions with respect to α on $[a,b]$, for each $\epsilon > 0$ there exists a choice S on $[a,b]$ such that for any S-fine Henstock partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \le i \le k\}$, we have

$$D\left(\sum_{i=1}^{k} F_n(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})),\ (APFHS)\int_a^b F_n(x)d\alpha\right) < \epsilon$$

for all $n \in N$. Let us show that $\{(APFHS)\int_a^b F_n(x)d\alpha\}\}$ is a Cauchy sequence in the complete space (E^1, D). Let $P = \{([x_{i-1}, x_i], \xi_i) : 1 \le i \le k\}$ be a $S-$ fine Henstock partition on $[a,b]$. Note that

$$\begin{aligned} D(&\sum_{i=1}^{k} F_n(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), \sum_{i=1}^{k} F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1}))) \\ &\le \sum_{i=1}^{k} D(F_n(\xi_i), F(\xi_i))(\alpha(b) - \alpha(a)). \end{aligned}$$

Since $\{F_n\}$ converges pointwise to F on $[a,b]$, for each ξ_i there exists a $K_i(\xi_i) \in N$ such that

$$D(F_n(\xi_i),\ F(\xi_i)) < \frac{\epsilon}{k(\alpha(b) - \alpha(a))}$$

for all $n \ge K_i(\xi_i)$. Set $N_1 = \max\{K_i(\xi_i) : 1 \le i \le k\}$. Then

$$D\left(\sum_{i=1}^{k} F_n(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), \sum_{i=1}^{k} F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})\right) < \epsilon$$

for all $n \ge N_1$. Thus we have

$$D\left(\sum_{i=1}^{k} F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})),\ (APFHS)\int_a^b F_n(x)\,d\alpha\right)$$

$$\leq D\left(\sum_{i=1}^{k} F(\xi_i)(\alpha(x_i)-\alpha(x_{i-1})), \sum_{i=1}^{k} F_n(\xi_i)(\alpha(x_i)-\alpha(x_{i-1})\right)$$
$$+ D\left(\sum_{i=1}^{k} F_n(\xi_i)(\alpha(x_i)-\alpha(x_{i-1})), (APFHS)\int_a^b F_n(x)\,d\alpha\right) < 2\epsilon$$

for all $n \geq N_1$. Therefore, for all $n, m \geq N_1$, we have

$$D\left((APFHS)\int_a^b F_m(x)\,d\alpha, (APFHS)\int_a^b F_n(x)\,d\alpha\right)$$
$$\leq D((APFHS)\int_a^b F_m(x)\,d\alpha, \sum_{i=1}^{k} F(\xi_i)(\alpha(x_i)-\alpha(x_{i-1}))$$
$$+ D(\sum_{i=1}^{k} F(\xi_i)(\alpha(x_i)-\alpha(x_{i-1})), (APFHS)\int_a^b F_n(x)\,d\alpha) < 4\epsilon.$$

It follows that $\{(APFHS)\int_a^b F_n(x)\,d\alpha\}$ is a Cauchy sequence in the complete space (E^1, D). Let $\lim_{n\to\infty}(APFHS)\int_a^b F_n(x)\,d\alpha = H \in E^1$. We show that $(APFHS)\int_a^b F(x)\,d\alpha = H$. By hypothesis, there exists $N_2 \geq N_1$ such that

$$D\left((APFHS)\int_a^b F_n(x)\,d\alpha, H\right) < \epsilon$$

for all $n \geq N_2$. Set $N = \max\{N_1, N_2\}$. For all $n \geq N$, we obtain

$$D\left(\sum_{i=1}^{k} F(\xi_i)(\alpha(x_i)-\alpha(x_{i-1})), H\right)$$
$$\leq D\left(\sum_{i=1}^{k} F(\xi_i)(\alpha(x_i)-\alpha(x_{i-1})), (APFHS)\int_a^b F_n(x)\,d\alpha\right)$$
$$+ D\left((APFHS)\int_a^b F_n(x)\,d\alpha, H\right) < 3\epsilon.$$

Hence, F is AP-Henstock-Stieltjes integrable with respect to α on $[a, b]$ and

$$(APFHS)\int_a^b F(x)\,d\alpha = \lim_{n\to\infty}(APFHS)\int_a^b F_n(x)\,d\alpha.$$

This completes the proof. □

THEOREM 2.6. *Let $F : [a,b] \to E^1$ be a bounded AP-Henstock-Stieltjes integrable function with respect to a sequence of increasing functions $\{\alpha_n\}$ on $[a,b]$ and $\{\alpha_n\}$ converges pointwise to α on $[a,b]$. If F is equi-integrable with respect to the sequence $\{\alpha_n\}$ on $[a,b]$, then F is AP-Henstock-Stieltjes integrable with respect to α on $[a,b]$ and*

$$(APFHS)\int_a^b F(x)\,d\alpha = \lim_{n\to\infty}(APFHS)\int_a^b F(x)\,d\alpha_n.$$

Proof. Since F is equi-integrable with respect to the sequence of increasing functions $\{\alpha_n\}$ on $[a,b]$, for every $\epsilon > 0$, there exists a choice S on $[a,b]$ such that for any $S-$fine Henstock partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \le i \le k\}$, we have

$$D\left(\sum_{i=1}^{k} F(\xi_i)(\alpha_n(x_i) - \alpha_n(x_{i-1})),\ (APFHS)\int_a^b F(x)\,d\alpha_n\right) < \epsilon$$

for all $n \in N$. Let $P = \{([x_{i-1}, x_i], \xi_i) : 1 \le i \le k\}$ be an $S-$fine Henstock partition on $[a,b]$. Since F is bounded on $[a,b]$, $\sup_{x\in[a,b]} D(F(x), 0)$ exists. Since $\{\alpha_n\}$ converges pointwise to α on $[a,b]$, for each $x_i \in [a,b]$, there exists a $K_i(x_i) \in N$ such that for all $n \ge K_i(x_i)$, we have

$$|\alpha_n(x_i) - \alpha(x_i)| < \frac{\epsilon}{k\sup_{x\in[a,b]} D(F(x), 0)}.$$

Now, we show that $\{\int_a^b F(x)\,d\alpha_n\}$ is a Cauchy sequence in the complete space (E^1, D). Set $N_1 = \max\{K_i(x_i) : 1 \le i \le k\}$. Then for $n \ge N_1$, we obtain

$$\begin{aligned}
&D\left(\sum_{i=1}^{k} F(\xi_i)(\alpha_n(x_i) - \alpha_n(x_{i-1})),\ \sum_{i=1}^{k} F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1}))\right)\\
&\le \sum_{i=1}^{k} D(F(\xi_i)(\alpha_n(x_i) - \alpha_n(x_{i-1})),\ F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})))\\
&= \sum_{i=1}^{k} \sup_{\lambda\in[0,1]} \max\{|F_\lambda^-(\xi_i)(\alpha_n(x_i) - \alpha_n(x_{i-1}) - \alpha(x_i) + \alpha(x_{i-1}))|,\\
&\qquad\qquad |F_\lambda^+(\xi_i)(\alpha_n(x_i) - \alpha_n(x_{i-1}) - \alpha(x_i) + \alpha(x_{i-1}))|\}\\
&\le \sum_{i=1}^{k} |\alpha_n(x_i) - \alpha_n(x_{i-1}) - \alpha(x_i) + \alpha(x_{i-1})|
\end{aligned}$$

$$\sup_{\lambda\in[0,1]} \max\{|F^-_\lambda(\xi_i)|, |F^+_\lambda(\xi_i)|\}$$
$$\leq k\frac{2\epsilon}{k\sup_{x\in[a,b]} D(F(x),0)} \sup_{x\in[a,b]} D(F(x),0) = 2\epsilon.$$

Hence, $\{\int_a^b F(x)\,d\alpha_n\}$ is a Cauchy sequence in the complete space (E^1, D). Let $\lim_{n\to\infty}\int_a^b F(x)\,d\alpha_n = K \in E^1$. We show that $K = \int_a^b F(x)\,d\alpha$. By hypothesis, there exists $N_2 \geq N_1$ such that

$$D\left((APFHS)\int_a^b F(x)\,d\alpha_n, K\right) < \epsilon$$

for all $n \geq N_2$. For all $n \geq N_2$, we obtain

$$\begin{aligned}
&D\left(\sum_{i=1}^k F(\xi_i)(\alpha(x_i)-\alpha(x_{i-1})), K\right)\\
&\leq D\left(\sum_{i=1}^k F(\xi_i)(\alpha(x_i)-\alpha(x_{i-1})), \sum_{i=1}^k F(\xi_i)(\alpha_n(x_i)-\alpha_n(x_{i-1}))\right)\\
&\quad + D\left(\sum_{i=1}^k F(\xi_i)(\alpha_n(x_i)-\alpha_n(x_{i-1})), (APFHS)\int_a^b F(x)\,d\alpha_n\right)\\
&\quad + D\left((APFHS)\int_a^b F(x)\,d\alpha_n, K\right) < 4\epsilon.
\end{aligned}$$

Hence, F is AP-Henstock-Stieltjes integrable with respect to α and $(APFHS)\int_a^b F(x)\,d\alpha = \lim_{n\to\infty}\int_a^b F(x)\,d\alpha_n$. □

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Department of Mathematics Education
Chungbuk National University
Cheongju 28644, Republic of Korea
E-mail: yoonjh@cbnu.ac.kr